Developing Key Competences by Mathematics Education

Lifelong Learning Programme

Concepts for initial teacher education
Deliverable 2.2

1 Theoretical basis: European Reference Framework for Key Competences

In 2006, the Council and the European Parliament voted to accept a European Reference Framework for Key Competences for Lifelong Learning. On a European level, the Framework records key competences of individuals that are necessary for self-realisation, occupation and active citizenship.

The eight key competences are:

- Communication in the mother tongue;
- Communication in foreign languages;
- Mathematical competence and basic competences in science and technology;
- Digital competence;
- Learning to learn;
- Social and civic competences;
- Sense of initiative and entrepreneurship;
- Cultural awareness and expression.

These defined key competences prove true for a successful life in a democratic European knowledge-based society. In all eight key competences *critical thinking, creativity, initiative, problem-solving, risk assessment, decision-taking, and constructive management* are an issue.
(see: http://www.britishcouncil.org/sites/britishcouncil.uk2/files/youth-in-action-keycomp-en.pdf)

2 The General Idea of the Project KeyCoMath

KeyCoMath tackles the challenge of implementing the “European Reference Framework of Key Competences” in mathematics education. The project develops, evaluates and disseminates ways to bring this Reference Framework to practice in school and to increase its impact on the educational system.

KeyCoMath focuses on the following six key competences an uses specific didactical approaches to support them in mathematics education:

Key Competence	Didactical approach
Mathematical competence	KeyCoMath promotes pupils' active, exploratory learning in open and rich situations to deepen their ability of mathematical thinking.
Communication in the mother tongue	KeyCoMath closely intertwines doing mathematics and communicating with others orally or in written form. Pupils are encouraged to talk about mathematics, to discuss ideas, to write down thoughts and reflections and to present results.
Digital competence	In KeyCoMath pupils work with learning environments that include digital media (e.g. spreadsheets, dynamic geometry, computer algebra). By working mathematically they should develop a confident, critical and reflective attitude towards ICT.
Learning to learn	KeyCoMath emphasises pupils' self-regulated and autonomous learning. Thus, they develop abilities to manage their learning – both individually and in groups –, to evaluate their work and to seek advice and support when appropriate.
Social competences	KeyCoMath fosters pupils' collaboration and communication. They cooperatively do mathematics, discuss ideas, present findings and have to understand different viewpoints to achieve common mathematical results.
Sense of initiative	KeyCoMath strengthens exploratory, inquiry-based learning in mathematics. Pupils are encouraged to be creative, proactive and to turn ideas into action. They develop abilities to plan, organise, and manage their work.

3 Strategies for initial teacher education

Initial teacher education is the basis of mathematics education in the future. Thus, the partners implement the didactical approaches of the table in section 2 in initial teacher education. They arrange seminars and lectures where university students learn how to support pupils' key competences.

- In seminars and lectures university students they get acquainted with general ideas and **theories of teaching and learning**, they are made familiar with the didactic **concepts for supporting key competences** and they discuss and reflect on educational processes.

- To **bridge the gap between theory and practice**, the university students learn by doing: They apply the general concepts to concrete situations of mathematics education and **design** corresponding **"learning/assessment scenarios"**. These "learning/assessment scenarios" should give pupils opportunities to work according to the didactical concept depicted in the table in section 2 and, thus, to develop a variety of key competences. Furthermore, they should provide feedback to the teacher and the learner on deficiencies and proficiency with respect to the key competences aimed at.

- Whenever possible, these activities are related to **practical courses** in school where the university students work with pupils.

- All participants **present, discuss and reflect** their **results and experiences** cooperatively in university courses.

- Finally, the **learning/assessment** scenarios are **refined** by the students on the basis of all experiences. Good learning/assessment scenarios that match the quality standards of KeyCoMath are **published** on the project website and/or in further publications. It is quite motivating for the students to know that their work is **embedded in a European project** and that their products are published on an international level.

All these efforts mainly aim at changing and developing the participants' attitudes and beliefs towards mathematics education and their role in teaching and learning processes.

Concrete examples for initial teacher education

At the project meeting in Nürnberg some first recommendations from the partners were given on the basis of previous experiences in mathematics courses at university.

- **Lectures:** For example, the University of Bergen uses the didactic concept "learning by teaching" in lectures. In each session two university students have prepared some contribution to the lecture. These two students present an aspect of the lecture for about 20 min. This forms an integral part of the lecture.

- **Tutorials:** Implementing a tutor system appears to be a good way to ensmallen big university lectures. In groups of about 20 persons the students
    - work mathematically on their level and make experiences when working mathematically, they experience mathematics as a process of experience, exploration, discussion, documentation, presentation,
    - reflect on activities, build bridge to general aims and standards, extract general ideas of mathematics education, formulate general didactic ideas.
    - apply these general didactic ideas to concrete topics on pupils' level, construct learning environments for pupils according to the general didactic concepts.

  The academics or tutors have the possibility to make video analyses and to stimulate the development of students' ideas.

- **Seminars:** Academics have a wide creative leeway to organize their seminars. For developing key competences, it is helpful for students to experience learning scenarios by themselves already at university. Their fellow students could be in a pupil's position and test these assignments.
  As an option, a new mathematical topic with a difficulty level between school and university mathematics can be approached. In the first step, the university students bring a phenomenon into question and clarify it ("Explore why 60°.") and in the second step, they transfer it into practical classes ("What do I do in the classroom?"). In this manner, a bridge between theory and practice can be built.

- **Practical studies:** It is very valuable to combine practical studies in school with seminars at university. In these seminars teaching at school can be planned and prepared cooperatively, learning/assessment scenarios can be designed in a process of common discussion and all experiences form the lessons in school can be discussed and reflected. These reflections processes are crucial for developing the students' beliefs on mathematics and mathematics education.

4 Specific Standards for Mathematical Practice

"Mathematical competence" is one element in the European Reference Framework for Key Competences for Lifelong Learning. But how can "mathematical competence" be described more concretely? This question arises in university lectures and seminars on competence-oriented approaches to mathematics education – in the framework of KeyCoMath and beyond. The following description is taken from "Standards for the Mathematical Practice" written down by the "Common Core State Standards Initiative" (see: http://www.corestandards.org/Math/Practice/)

Make Sense of Problems and Persevere in Solving them
Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

Reason Abstractly and Quantitatively
Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize* – to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents – and the ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

Construct Viable Arguments and Criticise the Reasoning of Others
Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness

of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

Model with Mathematics
Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

Use Appropriate Tools Strategically
Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

Attend to Precision
Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

Look For and Make Use of Structure
Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more,

or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7 × 8 equals the well remembered 7 × 5 + 7 × 3, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2 × 7 and the 9 as 2 + 7. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.

Look For and Express Regularity in Repeated Reasoning

Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation $(y - 2)/(x - 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

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