Lifelong
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# Developing Key Competences by Mathematics Education

Carolin Gehring, Volker Ulm (Ed.)

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## Online Material

All worksheets for pupils printed in this book and further material for mathematics education are available digitally at:

**www.KeyCoMath.eu**

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Lifelong Learning Programme

Developing by Key Competences Mathematics Education

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# Contents

# Introduction:
# The European Project KeyCoMath

Carolin Gehring, Volker Ulm

This book is a result of the project "KeyCoMath – Developing Key Competences by Mathematics Education". It has been funded with the support of the Lifelong Learning Programme of the European Union between 2013 and 2015.

## 1 General Aims

The project KeyCoMath aims to develop pupils' key competences in primary and secondary schools. Didactic concepts, teaching and learning material as well as corresponding assessment methods for mathematics education are developed, tested, evaluated and disseminated on the European level. KeyCoMath uses the power of initial and in-service teacher education to put innovative pedagogical and didactical approaches into practice.

## 2 Project Consortium

KeyCoMath is based on the cooperation of eight partners from eight European countries:

- University of Bayreuth (DE)
- Bulgarian Academy of Sciences (BG)
- University of South Bohemia (CZ)
- University of Bergen (NO)
- University of Cyprus (CY)
- University of Klagenfurt (AT)
- German Department of Education in South Tyrol (IT)
- School Rottenschwil (CH)

## 3 Key Competences

Key competences are necessary for all citizens for personal fulfilment, active citizenship, social inclusion and employability in today's knowledge society. The project KeyCoMath develops, implements, and evaluates ways of working according to the "European Reference Framework of Key Competences for Lifelong Learning" in mathematics education.

The activities focus on the following key competences (basic skills and transversal competences) in primary and secondary schools:

- Mathematical competence
- Communication in the mother tongue
- Digital competence
- Learning to learn
- Social competences
- Sense of initiative

Further background and details are described in the following sections.

## 4 Target Areas

The project aims to innovate mathematics education on the European level in four target areas.

- KeyCoMath aims at fundamental changes of *pupils' learning*. A shift towards more active, exploratory, self-regulated, autonomous, communicative and collaborative learning is intended. This way of doing mathematics helps to develop a broad variety of key competences.

- These changes of pupils' learning require and are based on changes of *teaching*. Teachers develop expertise to create learning environments in order for pupils to work in the intended way and to develop key competences.

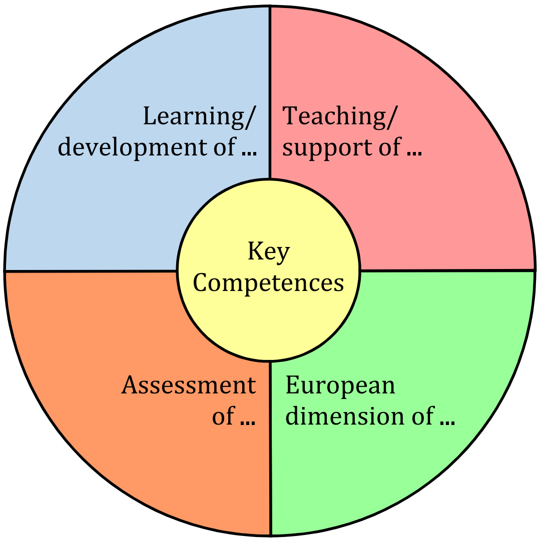
- *Assessment methods* that correspond to the competence-oriented approach are developed, tested and evaluated. Pupils work mathematically with open "learning/assessment scenarios", they write down their thoughts and findings, present and discuss results. Teachers are sensitized to use this information to diagnose pupils' competences, to adapt teaching to learners' needs and especially to support pupils with difficulties.

- KeyCoMath contributes to overcome the European problem of low-achievers. Furthermore, the partners identify and develop strategies for integrating the *European dimension* in teaching and learning mathematics.

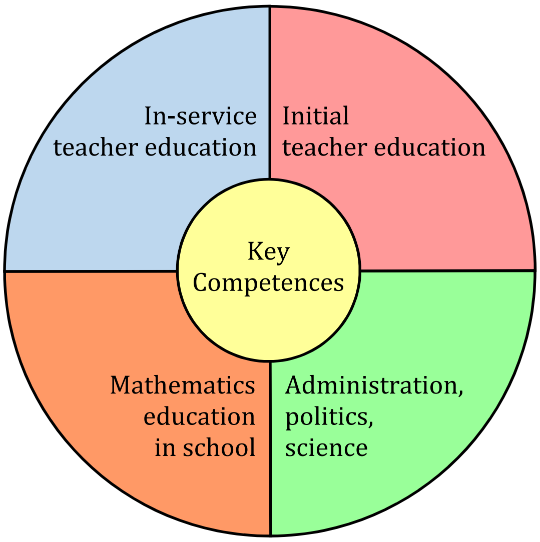
## 5 Fields of Activity

KeyCoMath operates in four fields of activity:

- The partners use the power of *in-service teacher education* for initiating innovations in mathematics education. Networks of teachers are intensively involved in the project. They are made familiar with didactic concepts for teaching, learning, and assessment to support pupils' key competences.
- *Initial teacher education* is the basis of future mathematics education. Therefore, the partners implement the didactical concepts in initial teacher education.
- Dissemination and exploitation strategies with publications of didactic concepts and concrete teaching/learning/assessment material in Bulgarian, Czech, English, German, Greek and Norwegian contribute to the improvement of *mathematics education in school* on the large scale in Europe.
- To enhance the impact on the level of educational systems, project results are distributed among decision-makers in *politics, administration* and *educational practice* as well as in the *scientific community* of mathematics education.




Target Areas of KeyCoMath




Fields of Activity of KeyCoMath

# Key Competences for Lifelong Learning and Mathematics Education: Concepts for Teaching and Learning Mathematics in Class

Demetra Pitta-Pantazi, Constantinos Christou, Maria Kattou, Marios Pittalis, Paraskevi Sophocleous

## 1 Introduction

This chapter aims to present a proposed framework which shows how basic skills and transversal competences can be developed by mathematics education. To this end, section two presents the Key Competences for Lifelong Learning according to the European Parliament and the Council of the European Union (2006). Section three provides the mathematical procedures and practices that students should develop according to various mathematics education organizations and researchers. Section four, having as a springboard the two previous sections, discusses a proposed framework which supports the development of students' key competences in mathematics. Finally, in section five we present assessment methods for students' key competences in mathematics.

## 2 Key Competences for Lifelong Learning

In the last ten years a lot of attention has been given at the European level to key competences for lifelong learning. A number of reports documented the importance of developing key competences at school in Europe (see European Commission/EACEA/Eurydice, 2012; Otten & Ohana, 2009). In addition, a significant number of research programs were conducted in the frame of the key competences (e.g., Q4i project: Quality for innovation in European schools, KeyCoNet project: Key Competence Network on School Education, EU-27: Key Competences and Teacher Education). At the end of 2006, these competences were defined by the European Parliament and the Council of the European Union (*Recommendation 2006/962/EC*), after a five year collaborative work between specialists to find a common framework of key competences for lifelong learning at the European level. Pepper (2011) characterized the development of these competences as 'an important policy imperative for EU member states' (Pepper, 2011, p. 335).
European Parliament and the Council of the European Union (2006) defined key competences as a multifunctional and transferable package of knowledge, skills and attitudes, which 'all individuals need for personal fulfillment and development, active citizenship, social inclusion and employment' (p. 13). In particular, European Parliament and the Council of the European Union (2006) identified eight key competences:

(1) Communication in the mother tongue
(2) Communication in foreign languages
(3) Mathematical competence and basic competences in science and technology
(4) Digital competence
(5) Learning to learn
(6) Social and civic competences
(7) Sense of initiative and entrepreneurship
(8) Cultural awareness and expression

These competences are equally important and interrelated. They are anticipated to be acquired both by students at the end of compulsory education, as well as by adults through a process of developing and updating their skills (European Parliament and Council of the European Union, 2006). Thus, initial education should offer all students the opportunities to develop key competences in a sufficient level that will equip them for adult and working life (European Parliament and Council of the European Union, 2006). In the following section, we provide a short description of the eight key competences:

"*Communication in the mother tongue*" requires an individual to have knowledge of words, terms, functional grammar and the functions of mother language. At the same time, this competence is connected to the acquisition of an individual's cognitive ability to interpret the world and relate to others. Moreover, it involves 'the skills to communicate both orally and in writing in a variety of communicative situations and to monitor and adapt their own communication to the requirements of the situation' (European Parliament and Council of the European Union, 2006, p. 14). This competence also includes the acquisition of positive attitude to critical and constructive dialogue, an acceptance of aesthetic qualities accompanied with an enthusiasm to try for them, and an interest in interaction with others (European Parliament and Council of the European Union, 2006; European Communities, 2007).

"*Communication in foreign languages*" requires an individual to have knowledge of the words, terms, functional grammar and structure of foreign languages. It includes skills for communication in foreign languages. These skills are comprehension of spoken messages, making conversations, reading and producing writing. Moreover, this competence involves 'the appreciation of cultural diversity, and an interest and curiosity in languages and intercultural communication' (European Parliament and Council of the European Union, 2006, p. 15).

"*Mathematical competence and basic competences in science and technology*" involves two essential competences: mathematics competence and science and technology competence.
'Knowledge in mathematics involves a sound knowledge of numbers, measures and structures, basic operations and basic mathematical presentations, an understanding of mathematical terms and concepts, and an awareness of the questions to which mathematics can offer answers. An individual should have the skills to apply basic mathematical principles and processes in everyday contexts at home and work, and to follow and assess chains of arguments. An individual should be able to reason mathematically, understand mathematical proof and communicate in mathematical language, and to use appropriate aids' (European Parliament and Council of the European Union, 2006, p. 15).
For science and technology competence, 'essential knowledge comprises the basic principles of the natural world, fundamental scientific concepts, principles and methods, technology and technological products and processes, as well as an understanding of the impact of science and technology on the natural world. Skills include the ability to use and handle technological tools and machines as well as scientific data to achieve a goal or to reach an evidence-based decision or conclusion' (European Parliament and Council of the European Union, 2006, p. 15).

"*Digital competence*" requires an individual to have knowledge of the 'nature, role and opportunities of technology in everyday contexts' (European Parliament and Council of the European Union, 2006, p. 16). This competence, also, includes 'the ability to search, collect and process information and use it in a critical and systematic way, assessing relevance and distinguishing the real from the virtual while recognizing the links' (European Parliament and Council of the European Union, 2006, p. 16).

"*Learning to learn*" requires 'an individual to know and understand his/her preferred learning strategies, the strengths and weaknesses of his/her skills and qualifications, and to be able to search for the education and training opportunities and guidance and/or support available. Learning to learn skills require firstly the acquisition of the fundamental basic skills such as literacy, numeracy and ICT skills that are necessary for further learning. Building on these skills, an individual should be able to access, gain, process and assimilate new knowledge and skills. Individuals should be able to organize their own learning, evaluate their own work, and to seek advice, information and support when appropriate. A positive attitude includes the motivation and confidence to pursue and succeed at learning throughout one's life' (European Parliament and Council of the European Union, 2006, p. 16).

"*Social and civic competences*" includes two essential competences: social competence and civic competence.
'Social competence involves knowledge of basic concepts relating to individuals, groups, work organizations, gender equality and non-discrimination, society and culture. The core skills of this competence include the ability to communicate constructively in different environments, to show tolerance, express and understand different viewpoints, to negotiate with the ability to create confidence, and to feel empathy.
Civic competence is based on knowledge of the concepts of democracy, justice, equality, citizenship, and civil rights, including how they are expressed in the Charter of Fundamental Rights of

the European Union and international declarations and how they are applied by various institutions at the local, regional, national, European and international levels. Skills for civic competence relate to the ability to engage effectively with others in the public domain, and to display solidarity and interest in solving problems affecting the local and wider community. This involves critical and creative reflection and constructive participation in community or neighborhood activities as well as decision-making at all levels, from local to national and European level, in particular through voting' (European Parliament and Council of the European Union, 2006, p. 17).

"*Sense of initiative and entrepreneurship*" requires individual to be able to turn ideas into action and to assess and take risks to materialize objectives. Individuals should also 'be aware of the ethical position of enterprises, and how they can be a force for good' (European Parliament and Council of the European Union, 2006, p. 17). In addition to this, 'an entrepreneurial attitude is characterized by initiative, pro-activity, independence and innovation in personal and social life, as much as at work' (European Parliament and Council of the European Union, 2006, p. 18).

Finally, "*Cultural awareness and expression*" 'includes an awareness of local, national and European cultural heritage and their place in the world. Skills relate to both appreciation and expression: the appreciation and enjoyment of works of art and performances as well as self-expression through a variety of media using one's innate capacities. Skills include also the ability to relate one's own creative and expressive points of view to the opinions of others and to identify and realize social and economic opportunities in cultural activity. A solid understanding of one's own culture and a sense of identity can be the basis for an open attitude towards and respect for diversity of cultural expression' (European Parliament and Council of the European Union, 2006, p. 18).

## 3 Mathematical Practices

In the mathematics education domain various organizations and educational authorities defined important mathematical procedures and practices that students should develop (National Council of Teachers of Mathematics, 2000; Standards for Mathematical Practice of the Common Core State Standards Initiative, 2011). In addition, the debate regarding the practical perspective and the mathematical rigor in developing mathematical curricular has been extensively discussed in various countries (Sullivan, 2011). In this framework, Ernest (2010) described the goals of the practical perspective and explained the importance of learning the mathematics adequate for general employment and functioning in society and mathematics necessary for various professional and industry groups. Furthermore, the Shape of the Australian Curriculum: Mathematics (Commonwealth of Australia, 2009) distinguished the practical and the specialized aspects of the mathematics curriculum and emphasized the need to educate students to be active, to interpret the world mathematically and use mathematics to make predictions as well as to take decisions regarding personal and financial priorities. The Victorian Department of Education and Early Childhood Development of the Government in Australia (2009) underlined the importance of developing strategic skills, such as knowing that mathematics might help, adapting mathematics to the context, knowing how accurate to be, and knowing if the result makes sense in context. It also made explicit twelve scaffolding practices that are appropriate to explore/make explicit what is known, challenge/extend students' mathematical thinking or demonstrate the use of a mathematical instrument. In particular, the twelve scaffolding practices refer to excavating, modeling, collaborating, guiding, convincing, noticing, focusing, probing, orienting, reflecting, extending and apprenticing.

Classroom mathematical practices focus on the taken-as-shared ways of reasoning, arguing, and symbolizing established while discussing particular mathematical ideas (Cobb, Stephan, McClain, & Gravemeijer, 2011). Mathematical practices describe the conditions under which students learn mathematics with deep conceptual understanding. The mathematical practices are not skill-based content that students can learn through direct teaching methods, but emerge over time from opportunities and experiences provided in mathematics classrooms (Hull, Balka, & Harbin-Miles, 2011). Thus, these opportunities and experiences should be coordinated by mathematics teachers and include challenging problems, collaborative groups and interactive discourse. The practices are interdependent, in other words are not developed in isolation from one another, and hence mathematics educators need to continually assess student progress on these practices in a holistic fashion. The intent is that these essential mathematical habits of mind and action pervade the curriculum and pedagogy of mathematics, in age-appropriate ways.

The Common Core State Standards for Mathematics described both Content Standards and Standards for Mathematical Practice. The Standards for Mathematical Practice of the Common Core State Standards Initiative (2011) described varieties of expertise that mathematics educators at all levels should seek to develop in their students. The theoretical foundation of the mathematical practices lies on important "processes and proficiencies" with longstanding importance in mathematics education. First, the mathematical practices are related to the NCTM (2000) process standards of problem solving, reasoning and proof, communication, representation, and connections. In addition, they take into consideration the mathematical proficiency of the National Research Council's report *Adding It Up*: adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately), and productive disposition. The following mathematical practices describe in a more mathematical-oriented framework the majority of the Australian scaffolding practices.

*Make sense of problems and persevere in solving them.*

The first practice involves the discussion, explanation and solution of a problem with multiple representations and in multiple ways, as well as students' persistence during the solution process. Students should be able to analyze givens, constraints, relationships, and goals, make conjectures about the solution and plan a solution, monitor and self evaluate their progress and change course if necessary. Thus, this practice consists of the following: working to make sense of a problem, making a plan, trying different approaches when the problem is difficult, solving a problem in more than one way, checking whether the solution makes sense and connecting mathematical ideas and representations to one another.

*Reason abstractly and quantitatively.*

The second practice relates to converting situations into symbols to appropriately solve problems as well as convert symbols into meaningful situations. In other words, it involves the appropriate use, interpretation and exploitation of the mathematical language. Thus, two salient abilities are involved; *decontextualizing* – to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, and *contextualizing*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Hence, this practice consists of the following: (a) representing problems and situations mathematically with numbers, words, pictures, symbols, gestures, tables and graphs, and (b) explaining the meaning of the numbers, words, pictures, symbols, gestures, tables and graphs.

*Construct viable arguments and critique the reasoning of others.*

The third practice involves justifying and explaining, with accurate language and vocabulary, why a solution is correct and comparing and contrasting various solution strategies and explaining the reasoning of others. Students may construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Students should be able to listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments. Summing up, this practice consists of explaining both what to do and why it works and working to make sense of others' mathematical thinking.

*Model with mathematics.*

Model with mathematics describes the ability to use a variety of models, symbolic representations and digital tools to demonstrate a solution to a problem, by identifying important quantities in a practical situation and mapping their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. In addition, students should be able to analyze those relationships mathematically to draw conclusions. Thus, this practice consists of applying mathematical ideas to real-world situations and using mathematical models such as graphs, drawings, tables, symbols, and diagrams to solve problems.

*Use appropriate tools strategically.*

This practice involves students' ability to combine various tools, including digital tools, to explore and solve problems as well as justifying their tool selection. Tools are used to deepen students' understanding of concepts. Students should be able to make decision regarding the appropriateness of the used tool. Thus, this practice consists of choosing appropriate tools for a problem and using mathematical tools correctly and efficiently.

*Attend to precision.*
Attend to precision relates to students' appropriate use of symbols, vocabulary and labeling to effectively communicate and exchange ideas. Students should be able to use clear definitions with others and in their own reasoning. This practice consists of communicating mathematical thinking clearly and precisely, and being accurate when someone counts, measures and calculates.

*Look for and make use of structure.*
This practice involves students' ability to see complex and complicated mathematical expressions as component parts. They can conceptualize complicated things, as single objects or as being composed of several objects. This practice consists of finding, extending, analyzing and creating patterns and using patterns and structures to solve problems.

*Look for and express regularity in repeated reasoning.*
This practice defines students' ability to discover deep, underlying relationships, find and explain subtle patterns and look for regularity in problem structures when solving mathematical tasks. This practice consists of using patterns and structures to create and explain rules and shortcuts, using properties, rules and shortcuts to solve problems and reflecting before, during and after solving a problem.

The above mathematical practices had a significant impact in the mathematics education community. For instance, the Mathematics Assessment Project, collaboration between the University of California, Berkeley and Shall Center at the University of Nottingham, developed an assessment material that brings into life and contributes in implementing in real classrooms the Common Core State Standards for Mathematical Practice.

# 4 Proposed Framework: Development of Concepts for Teaching and Learning in Class

## 4.1 Mathematics Education and Key Competences

Despite the importance of Key Competences and the great emphasis to their development at European level, little attention has been given to the role of mathematics education, as a main subject in the school system, in developing students' key competences. Based on this, the KeyCoMath project underlined that there is a need to infuse activities that promote the development of key competences in mathematics teaching and learning. Thus, the KeyCoMath project tackles the challenge of implementing the "European Reference Framework of Key Competences" (European Parliament and Council of the European Union, 2006) in mathematics education and focuses on the development, implementation and evaluation of the majority of the European key competences in mathematics. Specifically, at the beginning of the project, the following key competences were defined and elaborated in the framework of mathematics teaching and learning.

*Mathematical competence.* KeyCoMath argues that students' mathematical thinking can be developed and enhanced through an active, exploratory learning in open and rich situations.

*Communication in the mother tongue.* KeyCoMath closely intertwines doing mathematics and communicating with others orally or in written form. For this, KeyCoMath claims that students should be encouraged to talk about mathematics, to discuss ideas, to write down thoughts and reflections and to present results.

*Digital competence.* KeyCoMath supports that students should be engaged in learning environments that utilize digital media (e.g. spreadsheets, dynamic geometry, computer algebra). By working mathematically they develop a confident, critical and reflective attitude towards ICT.

*Learning to learn.* KeyCoMath recommends that emphasis should be given to students' self-regulated and autonomous learning. Thus, students should develop abilities to manage their learning (both individually and in groups), to evaluate their work and to seek advice and support when appropriate.

*Social competences.* KeyCoMath argues that teaching of mathematics should provide students opportunities to collaborate and communicate. They cooperatively do mathematics, discuss ideas, present findings and have to understand different viewpoints to achieve common mathematical results.

*Sense of initiative.* KeyCoMath supports that students should be encouraged to be creative, proactive and to turn ideas into action though exploratory, inquiry-based learning in mathematics.

Through this process students might develop abilities to plan, organize, and manage their work.

## 4.2 The Proposed Framework

In the following, we outline the way in which the KeyCoMath project investigated all the above Key Competences and Mathematical practices and reached a new framework for Key Competences that should be developed in the mathematics classroom. The purpose of the proposed framework is to synthesize the two strands of research in the domain of key competences, i.e., the European framework of key competences and the recent efforts in mathematics education to explicitly define important mathematical practices. Thus, in an attempt to provide a practical framework that defines mathematical practices which contribute in enhancing students' key competences, we suggest a five-pillar comprehensive description of mathematical practices and appropriate didactical approaches.

*Explore problems and persevere in solving them.*
We set off from the principle that one learns a lot of things in mathematics if he/she is given the opportunity to solve a lot of problems. Working on problems teaches students the importance of persistence when dealing with a mathematical problem and the importance to develop reasonable arguments. Through problem solving students learn to learn.
Through the exploration of problems students may also develop a sense of initiative since different approaches may be used. Students may be asked to solve a problem or carry out an assignment where limited or no specific directions are given. Such scenarios, force students to take their own initiative.

*Approaches*
- Use exploration activities: Mathematics discipline is a good environment for exploring. This gives students the opportunity to explore and learn by themselves.
- Offer plethora of problems: Students should be given the opportunity to solve different types of problems, for instance, easy and hard problems, problems that can be solved if you are working backwards. They should try to understand the meaning of problems, to analyze information, to identify the entry point for their solution.
- Engage students with problems with different solutions.
- Offer problems which can be solved with different approaches.
- Use open problems.
- Ask students to make conjectures: Students should try to make conjectures and plan approaches for solutions.
- Give students time: It is important to stress that learning to learn needs time, and time needs to be allocated to this purpose. Students should be given the time to reflect on activities and learn by themselves.
- Ask students to reflect on activities that they carried out: Through problem solving students should be asked to reflect on their own learning, monitor and evaluate their procedures and ask themselves whether their solution makes sense.

*Communicate in mother tongue, construct viable arguments and critique the reasoning of others.*
One of the main concerns of mathematics teaching should be the flexible transition from everyday language to the language used in mathematics:
- Each child has to have the possibility to express his/her own thoughts and feelings through a mathematical approach.
- It is a long journey to reach formal mathematics. This may start from early years when students are asked to add for example 3+2. In such cases students may be asked to tell a story which represents this mathematical sentence. Teachers should try to break the separation that exists between natural language and mathematics.
- When using mathematical language, students need to realize that they need to be precise and accurate. The aim will be to bring precision in their language. This is of great essence and necessary when students make a transition to more typical and formal mathematics.
- When referring to communication it needs to be stressed that this should be achieved both in written and spoken language.
- Communication in mother tongue should be used in the classroom for students to communicate and understand their teacher as well as their peers.

*Approaches*
- Ask students to produce a learning journal: Students may be asked to write a journal

about the mathematics they learn in the classroom.

- Require from students to produce a poster or presentation: Students should be given the opportunity to present their mathematical approaches, solutions, projects to others in the form of a presentation or a poster. Of course, such activities will also enhance students' social and communication skills as well as their mathematical skills.
- Engage students to activities that require mathematization of everyday life: Such activities aim to connect mathematical language to mother tongue and everyday life. Through the process of mathematization students are asked to represent a real-world situation symbolically (in mathematics language) by describing, identifying, formulating and visualizing the mathematical problem in their own way; and then moving back by making sense of the mathematical solution in terms of the real solution, including the limitations of the solution.
- Ask students to reformulate a mathematical idea: Students are asked to express the same mathematical idea in different ways.
- Offer students the possibility to work on projects: Students may develop communication in their mother tongue if they are working on their own projects.
- Ask students to communicate and defend mathematical ideas with the use of language, symbols, diagrams, actions.
- Ask students to listen and read carefully other people's arguments.
- Ask questions to clarify issues presented to students.
- Ask students to decide whether certain arguments make sense to them.

One may think that communication in a foreign language may be something that does not have a place in the mathematics classroom. However, this is far from the truth. Foreign language may be a way of describing the communication in another language which students are not familiar with. This key competence is necessary to be developed when students are asked to communicate with a programming language.

*Approaches*

- Learning to communicate with the programming language: Students may be asked to write or read something in programming language. Of course this may be done in various levels of education and in various levels of difficulty.

Social competence may be developed through the interaction and communication that students have in the classroom both with their peers and their teacher.

*Approaches*

- Ask students to listen, talk, write, understand, and communicate: Students should be given the opportunity to communicate in verbal and written form. They should learn how to listen carefully to other, to talk with precision and make themselves understood by others.
- Require from students to accept different solutions and respect arguments: Students must learn to respect the opinions and arguments provided by other people and also accept different solutions.
- Engage students in co-actions when communicating.
- Require from students accuracy in their communication: Insist on accuracy in various activities.
- Apply dialogic learning in classrooms: Teaching should involve the approach of dialogic learning.

*Model with mathematics, contextualizing and decontextualizing everyday situations.*

Students should be able to use mathematics to deal with problems that arise in everyday life, society and workplace. They should be able to model, contextualize and decontextualize everyday situations. For instance, in early years this may start with simply stating an equation to solve a mathematical problem while later students may have to construct more complex models to solve a problem which may involve situations such as decision making, system analysis and trouble shooting.

*Approaches*

- Ask students to apply their knowledge in real world problems.
- Require from students to make sense of quantities and relationships in problems.
- Ask students to use different types of models, representations, graphs, diagrams to represent various relationships among quantities, concepts, shapes. Create coherent representations that facilitate to solve a problem.

- Offer students the possibility to contextualize and decontextualize mathematical ideas. Be able to represent concrete situations symbolically and be able to understand and deal with them and the reverse, be able to understand symbols and translate them into real life concrete scenarios.
- Offer activities to students in order to become able to understand and translate abstract ideas into symbols and become able to use them.

*Use appropriate tools strategically.*

Students with mathematical proficiency should be able to make decisions regarding which tool they need or is appropriate to carry out a specific task. Tools may be simply paper and pencil, rulers, protractors, calculators, mathematics software. Nowadays, the ability to use digital tools strategically is one of the central competences that individuals need to develop.

Digital competence allows students to develop intuitions about various mathematical concepts. Students need to know and also use various mathematical programs. Furthermore, when referring to digital competence it is important to raise the issue that students need to learn how various machines work.

*Approaches*

- Offer students the possibility to decide which tools to use and how to use them, for example, tools, such as paper and pencil, calculator, ruler, digital technologies, protractor, compass.
- Train students to use any kind of software system: Asking students to work with any kind of software system often increases and improves students' mathematical understanding. This may allow students to better visualize, compare and predict results.
- Offer students the possibility to use modern software systems: Students must be given the opportunity to work with modern software systems.
- Engage students with virtual reality activities in mathematics: Students must also be given the opportunity to work in developing mathematical ideas with the use of virtual reality.

*Look for and make use of structure and generalizations, attend to precision.*

Students should be able to identify a pattern or structure in what they see in mathematics and also be able to find commonalities and reach generalizations or find shortcuts in procedures. In the process of doing these as well as in the products of their mathematical activities students should be able to communicate with precision and accuracy.

*Approaches*

- Ask students to look for patterns or structure.
- Require from students to recognize that quantities can be represented in different ways.
- Give students the opportunity to use patterns or structures to solve problems.
- Offer students the possibility to view complicated concepts either as single objects or part of compositions of several objects.
- Give students tasks where they need to notice repeated patterns, calculations or methods and look for general methods or shortcuts.
- Ask students to reflect and evaluate the reasonableness of results and make generalizations.
- Require from students to communicate precisely and with accuracy.
- Ask students to state the meaning of concepts, symbols, and carefully specify units of measure.
- Require from students to calculate accurately and efficiently, to carefully state explanations and provide accurate labels.

## 5 Key Competences and Assessment

In recent years great emphasis is given on the ways used to assess key competences. Gordon et al. (2009) found that the most of the EU member states implemented curricula that include competences and identified assessment as one of their most important components. In particular, assessment might give information about learning process, can lead to the development of key competences and may support consequently effective changes (European Commission, 2012). Indeed, 'The assessment of key competences or similar learning outcomes that emphasize not only knowledge but also skills and attitudes in relation to contexts intended as preparation for lifelong learning' (European Commission, 2012 p. 4). Despite the importance of the assessment, the European Commission/EACEA/Eurydice report (2012), declares that most countries use national tests to assess only three out of the eight key competences (communication in the mother tongue, communication in foreign languages and mathematical competence and basic competences in science and technology). In addition to this, the national assessments in the most countries focus on the basic skills (mother tongue and mathematics) and in some of them focus on the science, foreign languages and social and civic competences (European Commission/EACEA/ Eurydice, 2012). For this, there is a need to reveal assessment methods for the key competences.

The first step towards assessment is the operationalization of the key competences, as it has been done in previous sections (Gordon et al., 2009). By defining the key competences and the scope of the assessment, as well by discussing the learning outcomes might give an insight for the assessment criteria (Sadler, 1987; Wolf, 2001). Furthermore, by describing the characteristics of the assessment tasks and the analysis of their results in the learning process, students and teachers can identify and appraise performances.

KeyCoMath project proposed four characteristics that tasks for measuring students key competences should have:

(1) include important mathematical aspects without complexity,
(2) allow the emergence of different solutions or processes,
(3) encourage and request pupils' productions (drawings, reasons),
(4) demand reflections such as descriptions, explanations or reasons.

First, as the aim of the assessment tasks is to reveal solver's cognitive processes and results, a suitable task should be clearly specified, easy to understand, is comprehensibly formulated and it fits to the current learners' state of knowledge. A similar conception is mentioned by Harlen (2007) 'A clear definition of the domain being assessed is required, as is adherence to it' (p. 18).
Secondly, although the task is based on known facts it might offer opportunities to the solver to link various competencies, to think critically and to deal actively in a situation. The exploration of open situations offers opportunities to the solver to provide a variety of solutions or different ways of approaches and problem-solving strategies. These type of tasks allows pupils to work on their own pace and abilities, to develop mathematical competences on their individual level, forcing them to activate their initiative and autonomy.
Thirdly, the use of appropriate scenarios such as occupational and social contexts asks learners to operate with everyday life problems and situations and adapt their knowledge in different set of circumstances (European Centre for the Development of Vocational Training, 2010). Such contexts demands from the solver to exploit their social, civic and cultural awareness.
Finally, appropriate assessment task should encourage the actual use and level of language, enhancing the competence of communication. As Gallin and Ruf (1998) mentioned, the use of language enables thoughts to be clarified and helps a response to be elicited.

Other assessment methods with the potential to assess key competences are among others standardised tests, attitudinal questionnaires, performance-based assessment, and portfolio assessment, teacher, peer and self-assessment practices (Looney, 2011; Pepper, 2013), as follow:

- *Standardised tests*: they mainly used for the assessment of some of the key competences (communication, mathematical competence, competences in science and technology), due to the fact that these three competences can be directly linked to individual subjects (Eurydice, 2009). As the other competences are more general and are related to several subjects, difficult appropriate standardised test might be developed (Pepper, 2013).

- *Attitudinal questionnaires*: these questionnaires aimed at capture learners' attitudes for learning to learn and generally for assessing affective and metaffective domains. However there is a difficulty of separating cognitive and affective aspects of learning (Frederiksson & Hoskins, 2008). Additionally a discrepancy between what students answer and what they actually do appears (European Commission, 2012).

- *Performance-based assessment*: this type of assessment includes portfolios, reflective diaries, experiments, group work, play, presentations, interviews and role plays (Looney, 2011). The observation of such behaviors might provide reliable results. 'Variation in teachers' judgements within and between schools is nonetheless a risk…' (Pepper, 2013, p. 18).

Classroom observation and dialogue is considered as an appropriate method for the assessment of key competences, in contrast to questionnaires and tests (Pepper, 2013). An inquiry based learning environment might offers great opportunities to this direction (Gallin, 2012). Through students experimentation in combination with teachers' involvement several key competences may be revealed and consequently might be assessed. For instance, as students attempt to find one or more solutions to a given task, mathematical competence is inevitably engaged. A careful selection of a task might also engage the cultural awareness and expression competence as the given information or even the required ones might be connected with the social and economic opportunities in cultural activity.

Additionally, if a teacher provides students with digital and electronic means as facilitators of problem solving process digital competence may also be developed. The use of digital means, in combination with cooperative work, forces learners to depict the communication and social competences.

Exchanging thoughts among the learners, discussing ideas and solutions between teacher and students, keeping track of their thoughts, problems and findings, judging others solutions, advising their peers might develop and assess the competences of communication and learning to learn.

Moreover, the role of teachers is important to provide appropriate feedback on remarkable insights. Reflection and redirection of students' thoughts may enable them to be aware of their strengths and weaknesses and consequently to identify fruitful use of them (Gallin & Ruf, 1998).

## 6 Conclusions

Summing up, the importance of key competences for lifelong learning has been widely documented (e.g. Eurydice, 2012; Otten & Ohana, 2009). Based on the previous sections of this chapter, is obvious that mathematics education may achieve the goal of developing key competence in a sufficient level and thus be able to build a more qualified citizen for the needs of today's world. Citizens who are able to explore everyday problems with perseverance, who are able to communicate and collaborate effectively, construct viable arguments and judge the reasoning of others, who develop models, who are able to use tools appropriately and who look for and use structures and generalizations, are sufficiently equipped to confront contemporary challenges.

Besides the development of key competence, their assessment is of equal importance. The development and assessment of key competences should be directed linked in order to gain better results in education. In particular, by revealing the ways that lead to the enhancement of key competences, insights are given for the development of appropriate assessment criteria (Sadler, 1987; Wolf, 2001). Additionally, the assessment process might give information about the learning process, and thus lead to the development of key competences (European Commission, 2012).

Hence, the research program KeyCoMath contributed in two strands: first, by defining the key competences and by proposing appropriate didactical approaches for their development; secondly, by proposing assessment methods with the potential to assess key competences. In order to ensure that "key competences" are in the focus point of both researchers and educators, it is important to educate teachers on this topic and especially train them about the ways in which key competences can be interwoven with their curriculum. Therefore, further studies could focus on developing didactical concepts that aim to maximize the impact of key competences in the field of teacher education and/or ways to measure the effectiveness of such programs.

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# Key Competences for Lifelong Learning: Concepts for Initial Teacher Education

Mette Andresen, Helena Binterová, Carolin Gehring, Marta Herbst, Pavel Pech, Volker Ulm

## 1 Introduction

Initial teacher education is the first step putting a teacher's career on course (Terhart, 2004). Future teachers acquire professional knowledge as well as theoretical basic knowledge (e. g. in pedagogy and psychology) for the subsequent professional practice (Neuweg, 2004). They are learner and teacher at the same time; one might speak of a student's double role during the initial teacher education: "being supported in learning how to teach, and supporting pupils in how to learn." (Caena, 2014) To apply their newly acquired knowledge in practical classes will be a challenging task (Keller-Schneider & Hericks, 2011). This circumstance should usefully be reflected in the structuring of initial teacher education. As the parameters are set by the professional practice, it is mentally demanding to analyse, question and review theoretical concepts. The focus is set on the entity of a person, including their behaviors and attitudes, their knowledge and competences. (Caena, 2014) The development of competences not only plays an important role for the development of the pupils as future world citizens, but also for the teachers. Teacher students and teachers are required to gain competences and thus act as an inspiring example for their students. (OECD, 2011)
The project KeyCoMath tackles the challenge of implementing the "European Reference Framework of Key Competences" (see previous chapter) in mathematics education. The project develops means to put this Reference Framework into practice in school and to increase its impact on the educational system. Initial teacher education is the basis of future mathematics education. Thus, the partners implement didactical approaches for supporting key competences in initial teacher education. Teacher students should develop corresponding professional competences.

## 2 Didactical Approaches

The project activities in initial teacher education focus on the following six key competences and use specific didactical approaches to support them in mathematics education:

| **Key Competence** | **Didactical Approach** |
|---|---|
| Mathematical competence | KeyCoMath promotes pupils' active, exploratory learning in open and rich situations to deepen their ability of mathematical thinking. |
| Communication in the mother tongue | KeyCoMath closely intertwines doing mathematics and communicating with others orally or in written form. Pupils are encouraged to talk about mathematics, to discuss ideas, to write down thoughts and reflections, and to present results. |
| Digital competence | In KeyCoMath pupils work with learning environments that include digital media (e.g. spreadsheets, dynamic geometry, computer algebra). By working mathematically they should develop a confident, critical and reflective attitude towards ICT. |
| Learning to learn | KeyCoMath emphasizes pupils' self-regulated and autonomous learning. Thus, they develop abilities to manage their learning – both individually and in groups –, to evaluate their work, and to seek advice and support when appropriate. |
| Social competences | KeyCoMath fosters pupils' collaboration and communication. They do mathematics cooperatively, discuss ideas, present findings and have to understand different viewpoints to achieve mathematical results. |
| Sense of initiative | KeyCoMath strengthens exploratory, inquiry-based learning in mathematics. Pupils are encouraged to be creative, proactive, and to turn ideas into action. They develop abilities to plan, organize, and manage their work. |

## 3 Strategies for Initial Teacher Education

The universities participating in KeyCoMath implement the didactical approaches of the table in section 2 in initial teacher education. They arrange seminars and lectures where university students learn how to support pupils' key competences.

*Mathematical expertise*

Mathematical knowledge is a basic prerequisite for future mathematics teachers. Even if some elements of higher mathematics play no role in teachers' practices at work with pupils, some other elements still need to be included in teachers' education e.g. polynomials, analytic geometry, derivatives, integrals, lines and surfaces in space, etc. Nevertheless, one should look for connections and examples of the application of concepts whenever it is possible. Mathematics courses need to be set in such way that university students are able to gain an understanding of the role of school mathematics in contemporary mathematics science.

Furthermore, initial teacher education should enable teacher students to obtain a deeper understanding of basic elements of mathematics, such as the change between forms of representation or the appropriate use of the language of logic, e.g. words "and", "or", "if . . .then", "all", "some", etc.

*Experiencing mathematical processes*

In initial teacher education, attention should always be paid to ensure that university students gain (more) experience to do mathematics in the form of observation, comparison, experiments, attempt, analysis, analogy, generalization, synthesis, induction, and deduction.

*Theoretical background*

In seminars and lectures, university students are acquainted with general ideas and theories of teaching and learning, they are made familiar with the didactic concepts for supporting key competences and they discuss and reflect on educational processes.

*Subject-matter didactic competences*

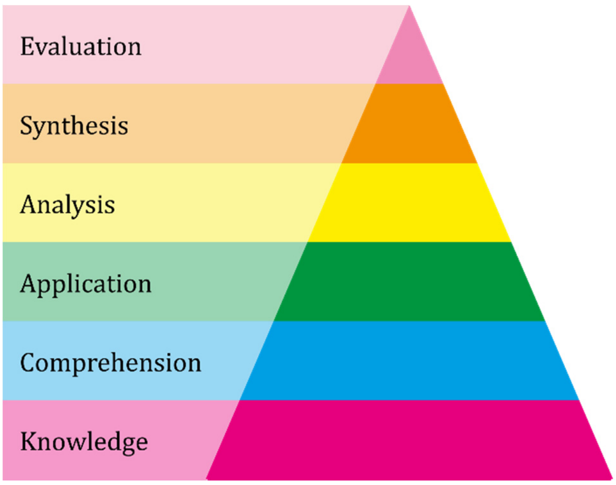
In addition to introduction to mathematics theory, student teachers need to know the entire subject matter of school mathematics as well as the didactic transformation of it. This means that in accordance to the goals of mathematics education, teachers need to deeply understand the topics of school mathematics, such as sets, measures and measurement, arithmetic, algebra, geometry, functions, probability etc. These topics are spirally developed throughout the grade levels. For this reason, it is very important for future teachers to know how to transform the content from these fields of mathematics into forms that are appropriate for learning in particular pupils' age levels.

*Learning/assessment scenarios as a tool for professional development*

To bridge the gap between theory and practice, the university students learn by doing: They apply the general concepts to specific situations of mathematics education and design corresponding "learning/assessment scenarios". These "scenarios" should support pupils in working according to the didactical concepts depicted in the table in section 2 and thus, to develop a variety of key competences. Furthermore, they should provide feedback for the teacher and the learner on deficiencies and proficiency with respect to the key competences aimed at.

*Taxonomy of Educational Objectives*

To develop such suitable tasks for "learning/assessment scenarios" is a challenge not only for teacher students, but also for teachers. It is very important for them to become aware of the degree of task difficulty. One classification scheme for categorising types of questions is provided by Benjamin Bloom with his taxonomy (see e.g. Alford, Herbert, Frangenheim 2006).




These categories can be described in the following way:

- *Knowledge:* Recalling information, discovering, observing, listing, locating, naming

- *Comprehension:* Understanding, translating, summarising, demonstrating, discussing
- *Application:* Using and applying knowledge, using problem solving methods, manipulating, designing, experimenting
- *Analysis:* Identifying and analysing patterns, organising ideas, recognizing trends
- *Synthesis:* Using old concepts to create new ideas, designing and inventing, composing, imagining, inferring, modifying, predicting, combining
- *Evaluation:* Assessing theories, comparing ideas, evaluating outcomes, solving, judging, recommending, rating

With the aid of this taxonomy, teacher students can reflect their prepared "learning/assessment scenario": Which level of task is it? Is it the intended one? How can the tasks be rephrased to target a higher or lower requirement?

*Corresponding methodical concepts*
For the practical realization of "learning/assessment scenarios" in class, methodological approaches are essential as well. Thus, in university courses e.g. the teaching principle "I-You-We", developed by Peter Gallin und Urs Ruf, is imparted (Gallin, Ruf 2014). It is an ideal entry design for cooperative learning: After a phase of individual analysis, the pupils exchange their view with a partner. In the plenary the results will be presented and reflected.

*Experiences in practice in school*
Whenever possible, these activities are related to practical courses in school where the university students work with pupils.

*Reflection in teams*
All participants present, discuss and reflect their "learning/assessment scenarios" with corresponding pedagogical and didactical ideas as well as their experiences from the work in school cooperatively in university courses.

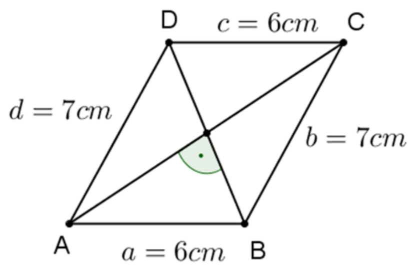
*Refinement of materials*
Finally, after all discussions and reflections, the "learning/assessment scenarios" are refined by the students on the basis of all experiences. Good learning/assessment scenarios that match the quality standards of KeyCoMath are published on the project website and/or in further publications. It is motivating for the students to know that their work is embedded in a European project and that their products are published on an international level.

# 4 Examples

The following examples illustrate the idea of discussing and reflecting tasks with respect to Blooms taxonomy.

## 4.1 Knowledge, comprehension – understanding, discussing

*Decide if the geometric shape, which is shown below, is a rhomboid. Explain your answer.*

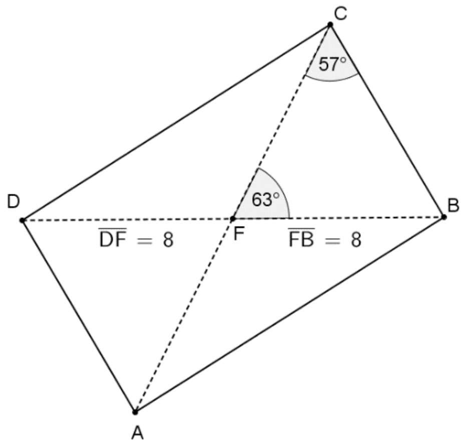



In this exercise, the pupils are intentionally given a visual representation of a non-existent geometric figure. Their task is to make a decision based on presented properties whether the presented figure is a rhomboid. The pupils are to apply the basic knowledge of diagonals and lengths of rhomboid's sides. Using the transfer of what has been learned and considering what they already know, based on this already existent mental representation, they should compare the presented nonverbal element to the incorrect data and discover the reason for non-existence of the figure. The correct answer is the discovery that the figure cannot be a rhomboid with the given lengths of the sides, since the diagonals bisect each other at a right angle.

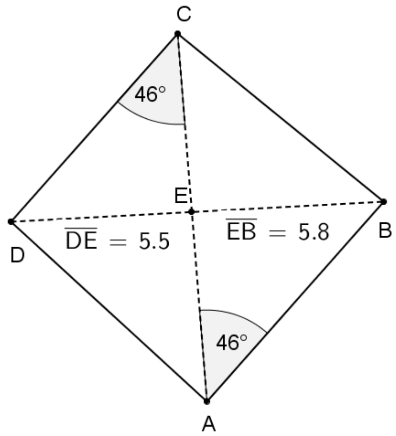
## 4.2 Comprehension, application – understanding, applying knowledge

*Decide and write down what kind of geometric object is in the picture below. (Square, triangle, rhombus, rhomboid, rectangle, tetrahedron, parallelogram, trapezium.)*

*a)*




*b)*




This exercise presents two visual representations of a quadrangle to the pupils. Pupils have to decide what kinds of quadrangles are in the picture. Stated visual representations are a rhomboid and a trapezium whose shapes are more evocative of a rectangle and a square. The pupils have to consider all of the important properties of these visual representations and compare them to their created mental representations of a rectangle and a square. This is where incorrect mental representations – based only on shapes of geometrical figures and not on characteristic properties like sizes of inner angles, diagonals bisecting each other, opposite sides being parallel – can become evident.

### 4.3 Analysis, synthesis – combination, organisation of ideas

*Draw and name a geometric shape with the following properties:*
*It is a geometric shape, whose opposite sides are parallel and congruent. Consecutive angles are different, but they are supplementary. Opposite angles are congruent. The diagonals of this shape bisect each other.*

In this exercise pupils are presented with a list of geometrical properties, based on which the pupils are to sketch a visual model corresponding to a geometric object that meets given conditions. The pupils are to use basic properties of a quadrangle concerning its inner angles, diagonals, and opposite sides. In this case of transfer of what has been learned, the pupils should create a correct mental representation of a geometrical object they have in mind and then express it visually. The correct solution is a sketch of a rhombus or a rhomboid.

## 5 Concrete examples for initial teacher education

At project meetings, the partners exchanged experiences from initial teacher education and gave recommendations for mathematics courses at university.

*Lectures*
For example, the University of Bergen, Norway, uses the didactic concept "learning by teaching" in lectures. In each session, two university students have to prepare a contribution to the lecture. These two students present an aspect of the lecture for about 20 minutes. This forms an integral part of the lecture.

*Tutorials*
Implementing a tutor system appears to be a good way to downsize big university lectures. In groups of about 20 people, the students

- work mathematically on their level and gain experiences when working mathematically, they experience mathematics as a process of experience, exploration, discussion, documentation, presentation,
- reflect on activities, build a bridge to general aims and standards, extract general ideas of mathematics education, formulate general didactic ideas,
- apply these general didactic ideas to specific topics on pupils' level, construct learning environments for pupils according to the general didactic concepts.

The academics or tutors have the possibility to make video analyses and to stimulate the development of students' ideas.

*Seminars*

Academics have a wide creative leeway to organize their seminars. For developing key competences, it is helpful for students to experience learning scenarios by themselves already at university. Their fellow students could take a pupil's position and test these assignments.

As an option, a new mathematical topic with a difficulty level between school and university mathematics can be approached. In the first step, the university students bring a phenomenon into question and clarify it and in the second step, they transfer it into practical classes ("What do I do in the classroom?"). In this manner, a bridge between theory and practice can be built.

A further example of a practically oriented seminar can be seen at the University of Bergen, Norway. Here, video material with real class situations is often used to be analysed.

*Practical studies*

Insights and initial professional experience make good sense in every respect and there is a variety of implementations.

The German and Czech project partners e.g. have the same procedure for their teacher students. At the beginning of their teacher training, university students visit classrooms as (guest) observers. Advanced teacher students complete a university-related internship at a school during one university term. It is very valuable in order to combine practical studies in school with seminars at university. In these seminars, teaching at school can be planned and prepared cooperatively, learning/assessment scenarios can be designed in a process of common discussion and all experiences from the lessons in school can be discussed and reflected with fellow students, the teachers at school and the university lecturer. These reflections and processes are crucial for developing the students' beliefs in mathematics and mathematics education.

The situation is different in South Tyrol, Italy. After their professional studies, students continue to study at university (1.5 days per week) extra-occupationally, whilst working as teachers at school.

*Other activities*

Teacher students of the University of South Bohemia regularly organize mathematical camps for pupils under the direction of members of the Faculty of Education. During their holidays, pupils can take part in mathematical activities such as geocaching, puzzling, paper folding etc. All of these activities are planned and supervised by university students. On the one hand, this is a good possibility for students to stay in contact with pupils and to get an impression of pupils' way of working and thinking. On the other hand, university students get the chance to try themselves out as learning guides and to test their self-prepared learning materials.

Summing up: All these efforts mainly aim to change and develop the teacher students' attitudes and beliefs towards mathematics education and their role in teaching and learning processes.

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# Concepts for In-Service Mathematics Teacher Education: Examples from Europe

Stefan Zehetmeier, Manfred Piok, Karin Höller, Petar Kenderov, Toni Chehlarova, Evgenia Sendova, Carolin Gehring, Volker Ulm

## 1 Introduction[1]

Teachers are considered to play a central role when addressing professional development programmes: "Teachers are necessarily at the center of reform, for they must carry out the demands of high standards in the classroom" (Garet, Porter, Desimone, Birman, & Yoon, 2001, p. 916). Ingvarson, Meiers, and Beavis (2005) sum up: "Professional development for teachers is now recognised as a vital component of policies to enhance the quality of teaching and learning in our schools. Consequently, there is increased interest in research that identifies features of effective professional learning" (p. 2).

In the past 20 years, newly emerging challenges for the teaching profession have resulted in an increased demand for corresponding (new) professional competences and an adequate framework (Posch, Rauch & Mayr 2009).

Until the 1990s, the role of teachers was mostly limited to offering pre-determined contents in a manner that is as clear and illustrative as possible, to ensuring discipline and assessing performance. This "static" culture of school-based teaching and learning has come under pressure in recent years, both with regard to student/teacher interaction and how classroom work is defined in substantive terms. Increasingly, the norms applicable at school are at variance with the wealth of extramural experiences children and adolescents gather. As a consequence, there are efforts to further develop and instil dynamic momentum into the static culture of teaching and learning.

The conditions of work and employment have changed: Graduates are increasingly expected to have multi-functional skills. They should be able to perform conceptual, planning and supervisory tasks. Higher demands are being placed in terms of a self-reliant design of work processes. And ultimately – due to the growing complexity of work settings – team work has gained increasing importance. Requirements such as these have created a demand for what has been called "dynamic" skills, for self-reliance, independent management and use of knowledge, and a sense of responsibility.

The socialisation which children and adolescents have experienced before they enrol at school has altered significantly in the past 30 years. Families tend to have fewer children, whilst the individual child plays a more central role. The relationship of parents and children has shifted to one that revolves around partnership. What is allowed and what is forbidden is more and more the outcome of a negotiating process between children and parents. It is with this background of experiences that children and adolescents come to school, projecting it onto this institution. As the prevailing work culture at school rests on a social model in which children and adolescents accept authoritarian decisions of adults, the resultant potential for conflict is inevitably huge.

Against this backdrop, teachers are facing challenges on a professional and human scale which are novel in many aspects. KeyCoMath aims at changes of pupils' learning. In particular, more active, exploratory, self-regulated, autonomous, communicative and collaborative learning is intended. Research on this kind of learning usually focuses on the students' level. Questions like these are raised: How can students' learning be well defined? How can we describe and explain students' progress and difficulties? However, these changes of pupils' learning are based on changes of teaching. Teachers develop expertise to arrange learning environments in order for pupils to work in the intended way and to develop key competences. Research on the teachers' level

[1] This introduction is based on Rauch, Zehetmeier & Erlacher (2014) and Zehetmeier, Rauch & Schuster (in preparation).

includes questions like: How can teachers be supported in activities dealing with this kind of learning? What do we learn from research when supporting teachers in implementing these learning activities? Teachers are considered to play a central role in planning, implementing, and researching professional development programmes.

This section provides some insights and examples, how various countries deal(t) with these issues. In particular, examples from Austria, Bulgaria, Germany and South Tyrol are provided.

## 2 Austria: The IMST Project[2]

Whereas in the last decades of the twentieth century many countries launched reform initiatives in mathematics and science instruction, or concerning teaching and teacher education in general, similar systemic steps in Austria did not happen. This led to a big gap between intended and implemented instruction, both in schools and at teacher education institutions. Although the promotion of students' understanding, problem solving, independent learning, etc. and the use of manifold forms of instruction and didactic approaches in mathematics and science instruction were regarded as important, teacher-centred instruction and application of routines dominated. Retrospectively, it seems clear that the educational system needed an external impulse, and this appeared in the late 1990s in the form of TIMSS 1995 and later PISA.

The initial impulse for the IMST project[3] in Austria came from the TIMSS achievement study in 1995. Whereas the results concerning the primary and the middle school were rather promising, the results of the Austrian high school students (grades 9 to 12 or 13), in particular with regard to the TIMSS advanced mathematics and physics achievement test, turned out to be disappointing (and evoked discussions on educational practice, research and policy, influenced by critical reports in the media). The ranking lists showed Austria as the last (advanced mathematics) and the last but one (advanced physics) of 16 nations (see e.g. Mullis et al. 1998, pp. 129, 189). This and other indicators showed that the teaching of mathematics and science in Austria needed a shift from "transmission" to "inquiry".

As in many other countries (see e.g. Prenzel and Ostermeier 2006), the responsible ministry reacted to the situation. In Austria, a national initiative with the aim to foster mathematics and science education was launched in 1998: the IMST project. Since then, this initiative has undergone several adaptions and is still running.

IMST was implemented in three steps:

1. The task of the *IMST research project* (1998–1999) was to analyse the situation of upper secondary mathematics and science teaching in Austria and to work out suggestions for its further development. This research identified a complex picture of diverse problematic influences on the status and quality of mathematics and science teaching: For example, mathematics education and related research was seen as poorly anchored at Austrian teacher education institutions. Subject experts dominated university teacher education, while other teacher education institutions showed a lack of research in mathematics education. Also, the overall structure showed a fragmented educational system consisting of lone fighters with a high level of (individual) autonomy and action, but little evidence of reflection and networking (Krainer, 2003; see summarized in Pegg & Krainer, 2008).

2. The *IMST$^2$ development project* (2000–2004) focused on the upper secondary level in response to the problems and findings described. In addition, it elaborated a proposal for a strategy plan for the ministry, aiming at improving the inquiry-based learning (IBL) of STEM in secondary schools. The two major tasks of IMST$^2$ were (a) the initiation, promotion, dissemination, networking and analysis of innovations in schools (and to some extent also in teacher education at university); and (b) recommendations for a support system for the quality development of mathematics, science and technology teaching. In order to take systemic steps to overcome the "fragmented educational system", a "learning system" (Krainer, 2005) approach was taken. It adopted enhanced reflection and networking as the basic intervention strategy to initiate and promote innovations at schools. Besides stressing the dimensions of reflection and networking, "innovation" and "working with teams" were two additional features. Teachers and schools defined

[2] This example is based on Krainer & Zehetmeier (2013) and Zehetmeier & Krainer (2013).
[3] IMST = originally, Innovations in Mathematics and Science Teaching (1998-1999); later, Innovations in Mathematics, Science, and Technology Teaching (2000-2009); since 2010 - motivated by adding German studies as one more subject - Innovations Make Schools Top.

their own starting point for innovations and were individually supported by researchers and project facilitators. The IMST$^2$ intervention built on teachers' strengths and aimed to make their work visible (e.g., by publishing teachers' reports on the website). Thus, teachers and schools retained ownership of their innovations. Another important feature of IMST$^2$ was the emphasis on supporting teams of teachers from a school.

3. The *IMST3 support system* (in four stages 2004–2006, 2007–2009, 2010–2012, 2013–2015, a fifth stage 2016–2019 is in preparation) started to implement parts of the strategy plan, among other ways by continuously broadening the focus to all school levels and to the kindergarten, and also to the subject German language (due to the poor results in PISA). The overall goal of IMST is to establish a *culture of innovation* and thus to strengthen the teaching of mathematics, information technology, natural sciences, technology, and related subjects in Austrian schools (see e.g. Krainer et al. 2009). Here, culture of innovation means starting from teachers' strengths, understanding teachers and schools as owners of their innovations, and regarding innovations as continuous processes that lead to a natural further development of practice, as opposed to singular events that replace an ineffective practice (for more details see e.g. Altrichter and Posch 1996; Krainer 2003).

For the future, the ministry expressed its intention to continue IMST. The overall goal is setting up and strengthening a culture of innovations in schools and classrooms, and anchoring this culture within the Austrian educational system.

## 3 Bulgaria: How can Teachers be Supported in Inquiry-Based Mathematics Education with Focus on Key Competences

### 3.1 Background

To create a class culture in which the teachers and the students could work as a research team using the ICT in support of the inquiry-based learning has been the goal of a long-term research in Bulgaria dating from the early 80's. The first attempts are related with the Research Group on Education (RGE) – having carried out an educational experiment launched by the Bulgarian Academy of Sciences and the Ministry of Education in 1979 (Sendov 1987, Sendov, Filimonov, Dicheva 1987; Sendova 2011). It comprised 29 pilot schools (2 % of the Bulgarian K-12 schools) and its main goal was to develop a new curriculum designed to make the use of computers one of its natural components. The guiding principles of RGE were learning by doing, guided discovery, and integrated school subjects. The RGE experiment ran for 12 years. Spreading the positive experience of RGE on a broader scale at the time turned out to be very difficult for various reasons – both economic and political. However, even with these isolated experiments the lessons learned were valuable – *the learners' and teachers' creativity alike can be enhanced when provided with appropriate environments.*

At a university level new courses were introduced, reflecting the need to prepare teachers working in the style of project based learning and guided discovery learning promoted in the RGE schools, e.g. at the Faculty of Mathematics and Informatics at Sofia University the curriculum for future mathematics teachers was enriched by the course *Teaching Mathematics in Laboratory Type Environment* (Nikolova, Sendova, 1995). After years of studying and reproducing very sophisticated mathematical facts these teachers-to-be experienced situations in which they could say: "Look at my construction!", "Can you prove my theorem?" (Denchev, Kovatcheva, Sendova 2012). Such a spirit of discovery was expected to be transferred later on to their students. Still, a negative tendency (noticed not only in Bulgaria but also in the most of Eastern Europe) was the slow decline of the educational system in the 90's and the early 2000's (Denchev, Kovatcheva, Sendova 2012).

### 3.2 New Life for Inquiry-Based Learning

With the advent of powerful ICT and specially designed educational software for mathematics explorations a way was opened for inquiry-based learning (IBL) in a number of European countries. Mathematics was an especially important domain of IBL thanks to the development and the dissemination of dynamic learning environments in which various experiments with mathematical objects could be performed leading to the formulation and verification of hypotheses of the learners themselves. Activities of this kind were the focal point of the involvement of the Bulgarian *KeyCoMath* team members in previous European projects, including *InnoMathEd, Fibonacci, DynaMath, Math2Earth, Mascil* and *Scientix* (Kenderov 2010; Kenderov, Sendova, Chehlarova, 2012).

Our work with in-service teachers is based on the understanding that for the teachers to be motivated they should experience the same intellectual pleasure we expect their students to undergo. The inquiry-based mathematics education (IBME) is promoted by our team at two levels – nationally and locally, in major regional centres. On a national level the promotion instruments are workshops, seminars and special sections of the national conferences organised by the Union of Bulgarian Mathematicians. On a local level IBME is promoted and supported by multiple training and presentation sessions organised in fifteen Bulgarian regions with the help of local boards (Chehlarova, Sendova 2012; Sendova, Chehlarova 2012).

### 3.3 Activities and Resources in Support of Inquiry Based Mathematics Education

The activities of the Bulgarian KeyCoMath research team in terms of organising events and developing educational resources embrace the four levels of the IBL:

Level 1 - Confirmation Inquiry, in which students confirm a principle through an activity whereby the results are known in advance;

Level 2 - Structured inquiry, in which students investigate a teacher-presented question through a prescribed procedure;

Level 3 - Guided inquiry, in which students investigate a teacher-presented question through a procedure they designed/selected themselves;

Level 4 - Open inquiry, in which students investigate a question they have formulated themselves through a procedure they designed themselves (Banchi, Bell 2008; Sendova 2014).

Here is a short description of such activities and resources:

*PD courses (from 2 to 128 hours)*

These courses are being organized by the Institute of Mathematics and Informatics of the Bulgarian Academy of Sciences (IMI-BAS) in the framework of European projects (InnoMathEd, Fibonacci, Mascil, KeyCoMath and Scientix), as well as by sections of the Union of Bulgarian Mathematicians (UBM), by the Ministry of Education and Science, by publishing houses for educational literature, and by PD centers. The main goal of the courses is in harmony with the most recent educational strategies for updating the math and science education in the EC countries: the development of key competences by implementing the inquiry based learning in integration with the world of work. These PD courses are based on a team work (of the lecturers and the participants alike) and implement educational models adaptable to various school settings. The crucial part of the courses is for the participants to experience different stages and levels of IBL with emphasis on key competences.

Each summer in the years 2014 and 2015 the KeyCoMath team has offered training courses for IBL for teachers in Bulgaria who are not involved directly in the project but want to learn and use the methods of IBL. The teachers work on pedagogical problems related with:

- reformulating math problems in IBL style so as to enhance the development of specific key competences;
- formulating their own math problems reflecting real-life situations, not solvable with the current math knowledge of the students but allowing for explorations by means of dynamic geometry models leading to a good enough approximation of the solution;
- studying and proposing methods for tackling problems which are unstructured, or whose solutions are insufficient or redundant;
- solving “traditional problems” with “non-traditional” data, for which the use of a computing device is necessary;
- applying game-design thinking so as to engage better the students in the problem solving;
- formulating more relevant evaluation criteria for the students’ achievements;
- project-based work with presentation of the results;
- assessment of learning resources in terms of formation and development of IBL skills and key competences (Kenderov, Sendova, Chehlarova 2014; Chehlarova, Sendova 2014).

The courses have two phases. In the first phase (3 days in the beginning of the summer) the teachers become acquainted with the dynamic geometry software GeoGebra as well as with plenty of examples of how to use dynamic scenarios in the IBL style. The participants of the courses are assigned projects as a “home-work”. In the second “follow-up” phase (with duration 1-2 days and conducted at the end of the summer) the participants present the advancement in the work on their project in front of the other participants in the course.

*PD events (seminars and workshops) in the frames of conferences*

The key feature of these events is that the teachers play an active role and act as partners in a research team – they share their good practices in oral or poster presentations (sometimes jointly with their students), work in groups on specific tasks and present their ideas to the rest of the participants. Typical examples include the *Scientix National Conference* within the National seminar *Inquiry Based Mathematics Education*, the *Dynamic Mathematics in Education* conference (Fig. 1), the seminars within the Spring conferences of UBM**,** the regional conferences organized by UBM sections, the International UNESCO workshop *QED* (Chehlarova 2012; Sendova 2015).


*Fig 1. The Scientix National Conference demonstrated good practices of teachers implementing IBL with emphasis on specific key competences*

The inquiry based learning, its connection with the world of work, good practices and problems directed at the development of specific key competences, have been the focus of our work with in-service teachers:

*Using specific learning scenarios in support of IBL*

A good repository of such resources is the *Virtual School Mathematics Laboratory* (Fig. 2, http://www.math.bas.bg/omi/cabinet/) being developed by IMI-BAS (Chehlarova, Gachev, Kenderov, Sendova, 2014), which contains over 800 scenarios with dynamic files transparent for the users. The way teachers are encouraged to use these resources is to stimulate students to behave like working mathematicians: to make experiments, to look for patterns, to make conjectures, to verify them experimentally, to apply "what-if" strategies so as to modify/generalize the problem, and even to use them as a preparation for a rigorous proof. To do this without leaving their comfort zone, the teachers enter the role of their students and experience the same type of activities during our courses, and when working on their own. They first use the dynamic files supporting the scenarios as a ground for explorations. The next step for them is to propose appropriate modification of the files for similar problems, or to use them as a model for creating one of their own from scratch. Thus, it would be quite natural for them to tell the students: "I don't know the answer, but I hope to find it together with you, thanks to YOUR efforts, to our joint efforts…"


*Fig. 2. Virtual School Mathematics Laboratory: dynamic files for the open problem on finding the locus of a regular m-gon inscribed in a regular n-gon*

*Building and developing competences necessary for the students to participate in new types of mathematics contests*

Examples are "Mathematics with a computer" and "Theme of the month" (Fig. 3, see Kenderov, Chehlarova 2014; Chehlarova, Kenderov 2015).

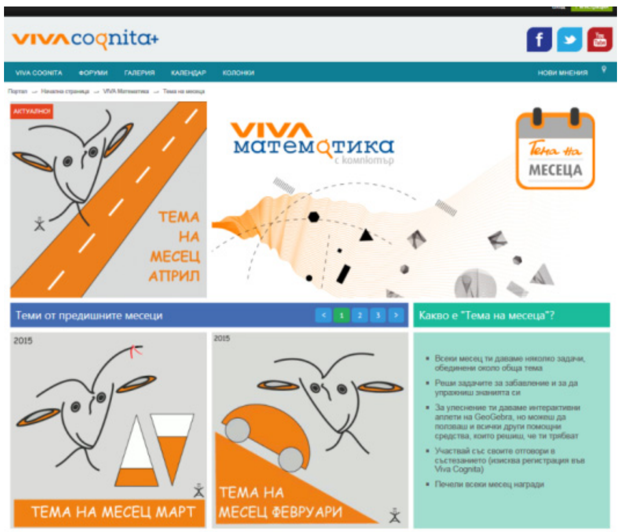


*Fig 3. "Theme of the month": an invitation for a long-term activity on a chain of math problems modeling real-life situations*

*Mathematics performances*

Events raise the awareness of the general public about the role of mathematics for enhancing children's scientific curiosity and endeavor to learn. The examples include: Performance at the History Museum in Stara Zagora, organized by the UBM section in the town, performances during the Researchers' Nights (2011-2015), Science festivals (in Italy, Romania, Greece). It is important to note that the teachers act as multipliers of the IBL ideas during these events as well – they participate with their students, and occasionally lead the performance.

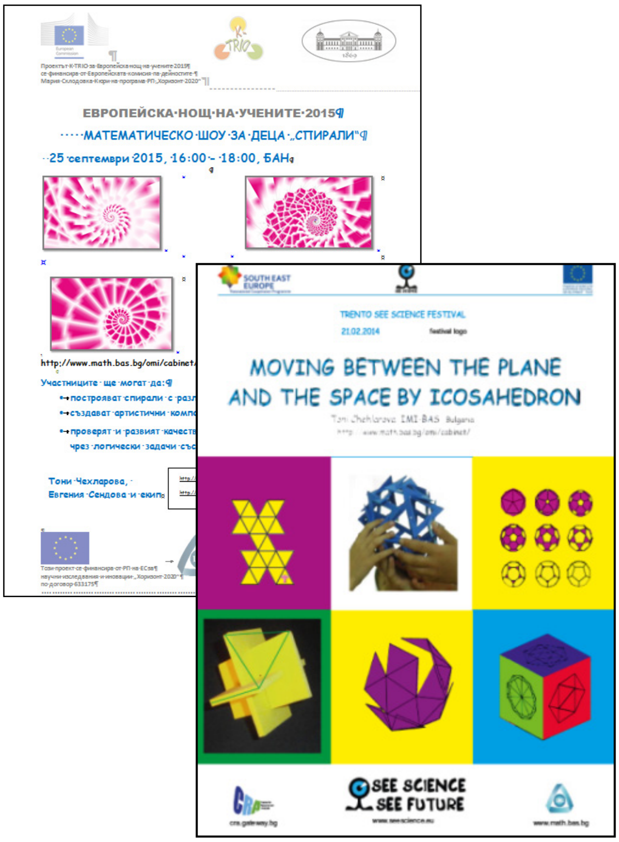


*Fig. 4. Posters for math performances within Science Fairs and Researchers' nights*

*Individual work with teachers*

It includes support for the development of a lesson, educational materials, mathematical festivals, course projects, peer reviews, and preparation of a pedagogical experiment.

*Activities in support of the Open Inquiry*

The fourth level of IBL (the Open Inquiry) has been promoted to reach teachers and students from all over the country with potential to do research in mathematics and informatics. This has been done in the frames of the High-School Students' Institute of Mathematics and Informatics (HSSI) where secondary school students work (under the supervision of a mentor) on their own projects, focused on the study of a given problem from Mathematics, Informatics and/or IT (Kenderov, Mushkarov, Parakozova 2015). The students deliver their results and findings, in written and oral form, in front of a jury and peers at two sessions – in January and April. A three week Summer School is organized for the involved teachers and the best achieving students. During the Summer Research Schools special seminars for teachers are organized. During these seminars experienced teachers share their good practices in mentoring students with high potential in doing research, and professional researchers in mathematics and informatics together with PhD students (HSSI alumni) deliver lectures on contemporary topics in these fields suggesting possible topics for future research projects.

## 3.4 The Main Achievements

A community of teachers who implement and spread the inquiry based learning of mathematics and informatics has been created. They participate in pedagogical experiments not only as a reality-proof of researchers but as members of a research team. These teachers implement, modify and develop from scratch educational resources in support of IBL, share their good practices at seminars, national and international conferences and in professional journals. Some of them organize public events at a school and regional level for popularizing the inquiry based mathematics education. Teachers are also key figures in organizing the new mathematics contests *Mathematics with a computer* and *Theme of the month*, in making them known to a broader audience.

The further goal is to reach faster and more effectively larger groups of other teachers and school students in acquiring the IBL approach with focus on the development of key competences.

The activities of the High School Institute of Mathematics and Informatics and the teachers results are encouraging. For its 15 years of existence it has proved that research potential of the students should be supported and developed starting at a very early age together with the development of other important competences including team work and presenting (orally and in written form) to various audiences. Some of the students' findings were on such a high level that they were published (and in some cases quoted by specialists) in mathematics research journals. The High School Student's Institute plays an important role with respect to the dissemination of IBL because the participating students (and their mentors) are coming from different towns in the country. Another positive effect is that a number of HSSI alumni act as mentors (virtually and face-to-face), thus passing the torch on to the next generation of young researchers.

## 4 Germany: Multipliers Concept for the Urban Network of Primary Schools

The University of Bayreuth, Germany, developed a practice-based multipliers concept for an urban network of primary schools with the aim to develop mathematics education and to support pupils' key competences. The idea is implemented on a local scale and addresses primary school teachers of the city Augsburg. The concept is realised within the framework of the European project KeyCoMath. 28 primary schools are taking part in a face-to-face professional development.

### 4.1 Involved Parties

On the urban scale, the University of Bayreuth coordinates, organises and supervises activities for the professional training of primary school teachers. It works directly with a group of qualified maths teachers as coaches for teachers' professional development. These teachers with their advisory function form a link between university and school and between science and practice, because they are responsible for appointed mathematical tutors from primary schools in the entire city.

### 4.2 Organisation

The Chair of Mathematics and Didactic of Mathematics is regularly in contact with the local education authority and with nine dedicated teachers who were assigned to become advisors due to their high professional expertise and their innovation capacity. Groups of two or three teacher-advisors guide 28 primary schools in the entire city. Therefore, a classification into four school groups according to location (north, south, east and west) has been assigned. Every primary school appoints at least two mathematical tutors – one responsible for the first and second form, the other for the third and fourth form. Hence, one school group consists of a minimum of twelve mathematical tutors who will further introduce/present newly gained acquaintances to their local colleagues.

### 4.3 Functions

The team at the University of Bayreuth has several duties: they keep in touch and make all arrangements with the local education authority. Teacher-advisors are appointed and meetings are summoned. They provide an academic support for schools and make funds as well as learning and teaching materials available. Above all, they operate a hosting platform for the information and material exchange. The nine mathematical teacher-advisors meet the university team regularly and they organise and guide school group meetings. There, they assist the mathematical tutors in planning teaching units. The tutors take an active part in the meetings in their assigned school group. They prepare and realise planned mathematical lessons and document teaching approaches.

### 4.4 Activities

Once a year, a major event for all involved primary school teachers takes place at the university. Such a meeting consists of a lecture to a superordinate topic like developing key competences by mathematics education. The attending teachers receive theoretical input and adopt it afterwards in workshops. The inclusion of pupils' utterances and solution processes conditioned by the study of pupils' written exercises, call logs or video shots demonstrates how children operate in a specific lesson's sequence.
In addition, every local primary school has the possibility to book in-house advanced training courses for its teaching staff according to the requirements. Normally, four school group meetings per school year are organised by the teacher-advisors. For these meetings, the mathematical tutors have to prepare the following: They previously realise a teaching unit in their own class and collect pupils' approaches. Above all, the members of a school group organise joint visits to classes, compare their varying approaches of teaching on the same specific mathematical topic, together reflect on one's own work and exchange their ready-made experiences, whereby mathematics education can be refined.

The focus is not solely set on spreading teaching materials and on giving recommendations for lessons. The objective of the project lies to a great degree on developing the level of inner convictions and beliefs teachers have of their subject, on teaching and learning processes, and their role in the lessons. These play a major role for the planning and implementation of lessons (Blömeke et al. 2010; Kunter et al. 2011).

### 4.5 Experience

This multipliers concept for the urban network of primary schools has proven itself over years and will be continued in future.
A coordinator says: *"It is encouraging to witness how mathematics education progresses at many primary schools in Augsburg, how teachers set out together to develop good mathematical assignments and how they bring teaching concepts of the common retrospect into question and thus improve them. By observing the children, I recognized their growing interest for mathematical issues, their creative dealing with numbers, patterns and structures, a greater openness for mathematical observations of everyday life and an increasingly safer handling with basic mathematical skills and abilities. The cooperation with school administration (education authority, administration) is successful because of good personal contacts."*

The involved primary teachers work as multipliers during their leisure time, including an additional effort in their daily workload, which is unfortunately neither compensated nor squared with class hour reduction, but which demonstrates the commitment of the participating teachers – since *"Staying still means taking a step back"* (a teacher advisor).

Concerning the observation of lessons, it is not always easy to find a volunteer willing to present the own teaching approach to colleagues. However, this risk is effectively prevented by the offer of a joint preparation of classes. *"I can work through exciting new ideas and discuss problems with motivated colleagues. Instead of sitting alone in my room I have many like-minded people around me."* (a participating teacher) and *"At the moment it's the only opportunity to get a direct look at the work of my colleagues. I can challenge myself with the implementation of the curriculum and at the same time receive inspiration through the teachers and their contact with students. Everyone learns from each other in a relaxed atmosphere without the pressure of being judged."* (a teacher who is on parental leave)

## 5 South Tyrol: Cooperative Design of Learning Environments

In the framework of KeyCoMath the German Department of Education in South Tyrol organised a teacher training for the development of competence-focused tasks for mathematics education at secondary schools. In a series of meetings the teachers were made acquainted with pedagogical and didactical theories concerning

- open questions,
- networking,
- individual learning,
- dialogic learning and
- assessment.

The participants, working in teams, developed tasks for their own lessons. These were tested in the classes and then, after a common reflection, optimised. The tasks focused on the acquisition of mathematical competences, but also on communicative, social and digital competences. The best practice examples of the tasks were published in a printed book „Tasks for competence-focused teaching of mathematics“ (Höller, Ulm 2015) and on the website www.KeyCoMath.eu.

*Strategies for Teachers' Professional Development*
Particularly valuable was the common work on tasks in the team of teachers – from the conception to the reflection and optimisation. Therefore, the planning of further training sessions will not only include contents and methods, but also ways of cooperative learning. The cooperation of the participants will take place as:

- teaching development in peer groups,
- common learning with mutual observation,
- exchanging views through e-learning platforms.

The focus lies on the development of teaching by the common design of learning environments for students. Moreover, the issue of evaluation is included.

These basic strategies for teachers' professional development are shown by the following diagram:

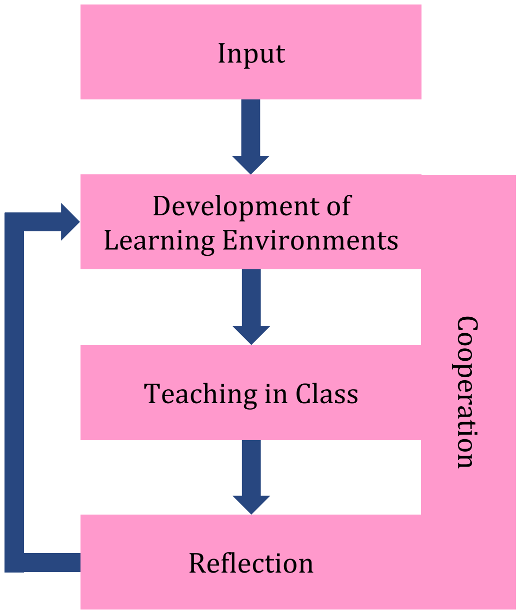


*Fig. 5. Teachers' professional development by working with learning environments*

## 6 Conclusion

In the project KeyCoMath partners from eight European countries developed concepts for innovating mathematics education. The focus lay on supporting students' key competences. Although the approaches in the different countries had to consider national frameworks, a common pattern of the strategies can be identified.

*Aiming at Teachers*
The key people for innovations in school are the teachers. Their beliefs, motivation and professional expertise are crucial for everyday teaching and learning. Thus, KeyCoMath focused on the teachers' professional development.

*Networks of Schools*
Since learning is a social process, regional networks of schools were established. They offered frameworks for teachers' exchange of experience and for their cooperative learning.

*Coaches for Teachers' Professional Development*
The regional school networks were led by a coach or a team of coaches who could be e.g. very experienced teachers, teacher educators or scientists. The teachers were made familiar with general didactical and pedagogical concepts. They related these ideas to their daily work at school; they designed learning environments for their students, and used them in their classes. The teachers presented, discussed and reflected their experiences cooperatively in their network of schools guided by their coach.

*Aiming at the Meta-Level*
Initiatives for substantial innovations of the educational system should aim at the meta-level of teachers' attitudes and beliefs. This concerns e.g. the role of the teacher, the role of the students, the nature of the subjects and general aims of education.

*Development of Learning Environments*
To bridge the gap between theory and practice, teachers individually and cooperatively developed learning environments for their students, worked with them in class, optimized them on the basis of all experiences, and exchanged and discussed them in their school network. Thus, by designing and working with specific learning environments teachers became acquainted with general pedagogical ideas. Learning environments function as tools for systemic teachers' professional development.

*Areas of Activity*
Participating schools and teachers should concentrate on one or a few areas of activity, e.g. exploratory learning, promoting students' cooperation, cumulative learning or fostering key competences. Such bounded fields of activity enable teachers to begin with substantial changes without the risk of losing their professional competence in class.

*Universities as Innovation Centres*
In these processes teachers and coaches received guidance and advice from universities. They served as innovation centres for teacher education. They provided regular and systematic in-service teacher education offers and coached the coaches.

*(Inter-)National Teacher Education*
Teachers and teacher educators were given the opportunity to exchange experiences with colleagues and to participate in professional development offers on a regional or a national level. They understood that problems and necessities of innovations have systemic character and concern the fundaments of education far beyond their own professional sphere. Moreover, they received ideas for innovation activities from a large community.

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# Learning Environments: A Key to Teachers' Professional Development

Carolin Gehring, Volker Ulm

## 1 Introduction

As it was described in the previous chapters, a core element of the project KeyCoMath was working with learning environments. Teachers and teacher students were made familiar with general didactical and pedagogical concepts for competence-oriented mathematics education. They related these ideas to practice by developing learning environments in teams. The teachers used them in their classes, the teacher students worked with them in practical courses in school. All experiences were cooperatively reflected and discussed afterwards. This strategy for teachers' professional development should closely intertwine pedagogical and didactical theories with practice in school. It should help teachers and teacher students to develop attitudes and beliefs for competence-orientated mathematics education.

## 2 A Model of Teaching and Learning

Teaching and learning are very complex processes. The model of "learning environments" shown in Fig. 1 simplifies reality – as any model does. However, basic structures of teaching and learning processes are revealed.
According to constructivist points of view, a teacher cannot put knowledge directly into the learners' heads. The *learning environment* is the essential link between teacher and learner. This notion includes five components: the *tasks* for the learner to work with the *content*, the *method* of teaching and learning, the arrangement of *media*, and the social situation with the teacher and other learners as *partners* for learning.
It is part of the teacher's professional expertise to design learning environments. Thus, he offers a basis for the learner's working. This allows the teacher to receive feedback about both the learner and the learning environment.

This model is based on and extends the didactical concepts of "substantial learning environments" by Wittmann (1995, 2001) or "strong learning environments" by Dubs (1995).
Sometimes "learning environments" are synonymously called "learning scenarios".
On the one hand, this model of learning environments shows that a teacher cannot enforce or steer students' learning directly. Limitations of a teacher's influence on students' learning are

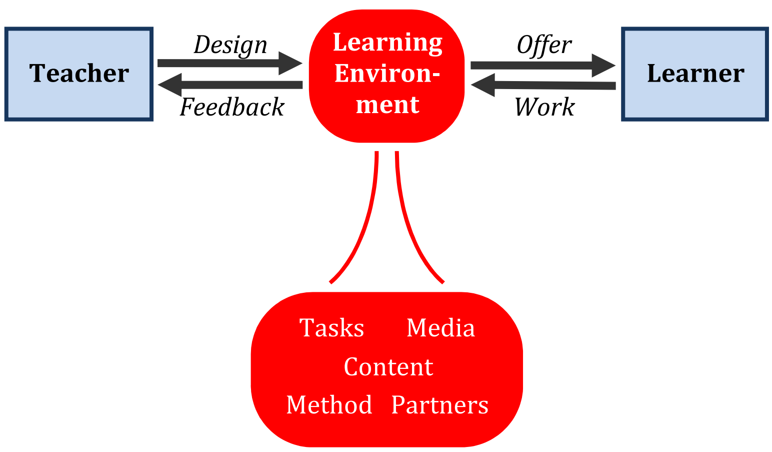


*Fig. 1: Model of learning environments*

shown. This might be disappointing or even frustrating for teachers.

On the other hand, interpreted positively, the model points out that it is a teacher's task to design learning environments in order to initiate and foster the students' learning and to use the feedback for further diagnosis and supporting activities.

## 3 Examples from Practice

In the project KeyCoMath teachers and teacher students designed learning environments, used them in class and reflected experiences cooperatively. These activities should initiate and support processes of teachers' professional development as it is described in section 1.

In the following chapters of this book, ten learning environments which result from these processes are presented. They give an impression of mathematics education that focuses on the development of key competences. However, with respect to the model in Fig. 1 it must be kept in mind that in the strict sense only "tasks" are published in the following sections.

They can unfold their potential for competence-oriented mathematics education only when they are embedded in appropriate learning environments with suitable methodical concepts. To illustrate this, the authors of the following chapters give some insight into the way they worked with the tasks. They describe the learning environments which they created for their students, and they report on results from practice.

Further examples from the project work are available on the website: www.KeyCoMath.eu

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# Lengths of Days over the Course of the Year

## Modelling with the Sine Function

Julian Eichbichler

### Intention

In this learning environment, the pupils should not only become acquainted with the course of the trigonometric functions but also learn about the influence of parameters on amplitude, period duration and phase shift. This is achieved through the use of explicit examples. After this, the motions of the moon in relation to the earth can be taken a look at: How are the seasons created? How does the length of day change over the course of the year when I am at North or South Pole? Which day is the longest/shortest? Why?

### Background of Subject Matter

The change in the length of day of the home town should be presented graphically and be approximated with an adequate curve (regression model). This function will be analyzed in different tasks. Through this, the pupils are acquainted with the general sine function and learn about the influence of the different parameters. Furthermore, they should be enabled to understand and question phenomena in daily life. For this topic, about four lessons are recommended.

The pupils should already be familiar with the trigonometric functions on the unit circle. Thus, they should have knowledge of the function $f(x) = \sin x$ and its co-domain, graph and periodicity.

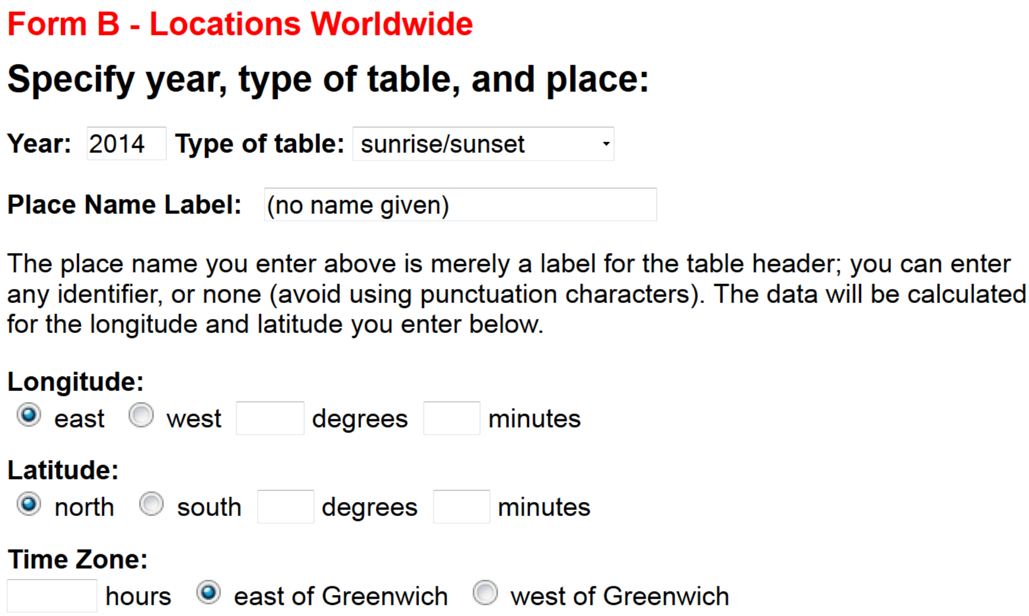
The task "Identify the change in the length of day over the course of the year" is not immediately connected to trigonometric functions by the pupil. This is only possible through the regression model. As a result, the pupils become familiar with the significance of the parameters in the function term of the general sine function $f(x) = a \cdot \sin(bx + c) + d$.

## Methodical Advice

On the website

http://aa.usno.navy.mil/data/docs/RS_OneYear.php

it is possible to calculate the exact time of sunrise and sunset for the home towns of the pupils.



**Form B - Locations Worldwide**

**Specify year, type of table, and place:**

**Year:** 2014 **Type of table:** sunrise/sunset

**Place Name Label:** (no name given)

The place name you enter above is merely a label for the table header; you can enter any identifier, or none (avoid using punctuation characters). The data will be calculated for the longitude and latitude you enter below.

**Longitude:**
east west degrees minutes

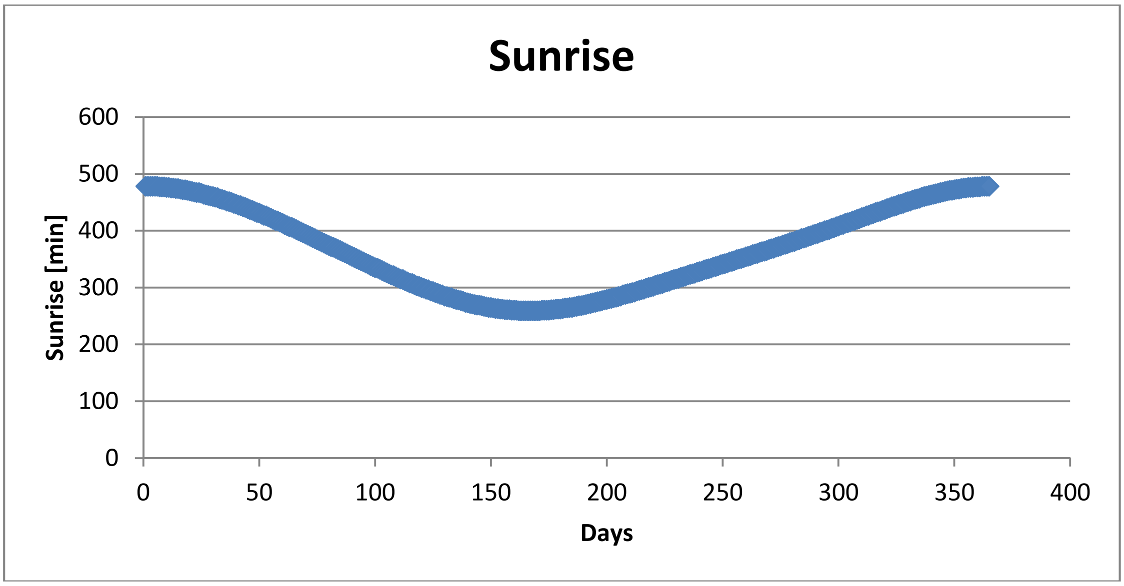
**Latitude:**
north south degrees minutes

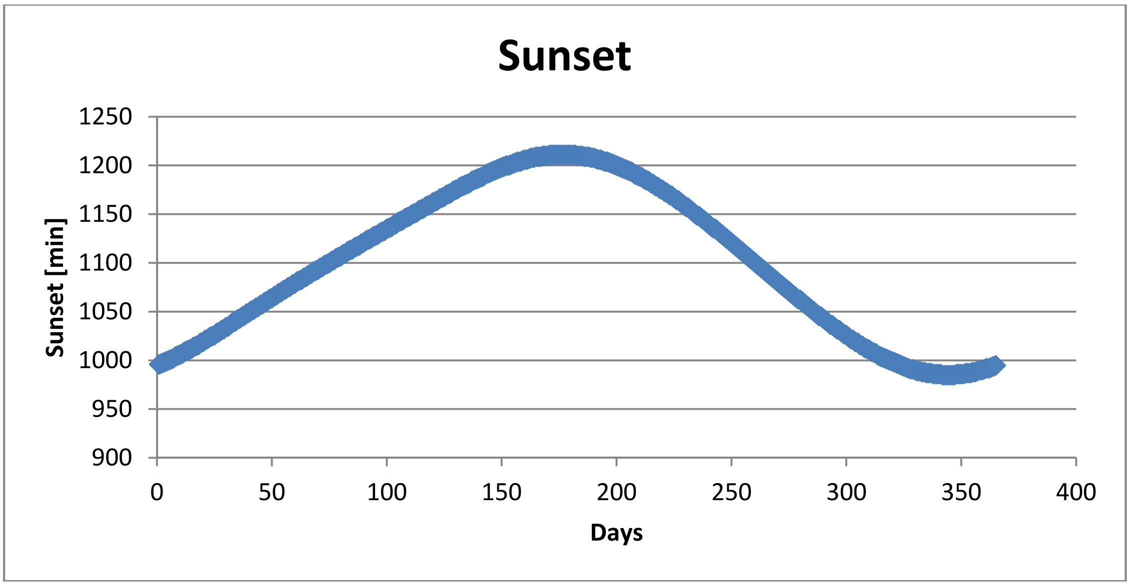
**Time Zone:**
hours east of Greenwich west of Greenwich

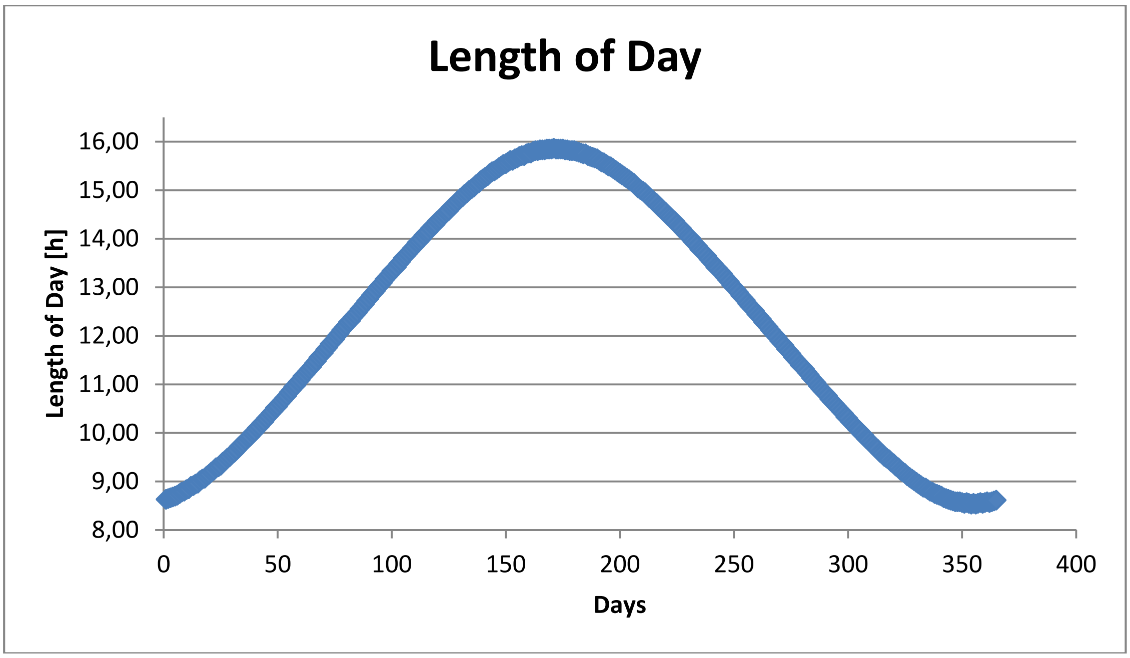
Firstly, the longitude and latitude of the home town have to be discovered. Furthermore, the time zone has to be identified and its influence explained.

On the website, there are instructions about the import of collected data to Excel.

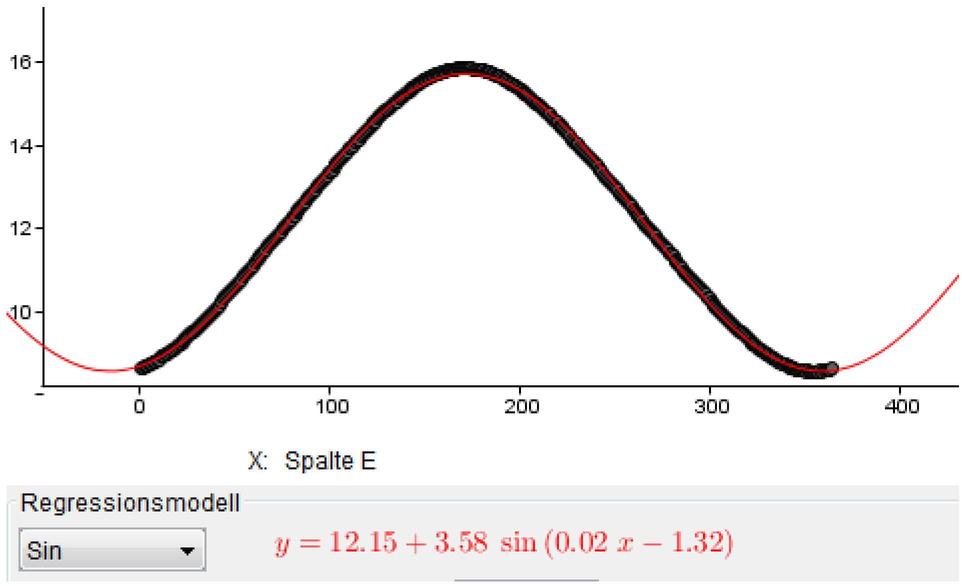
In class, the following graphs were found:

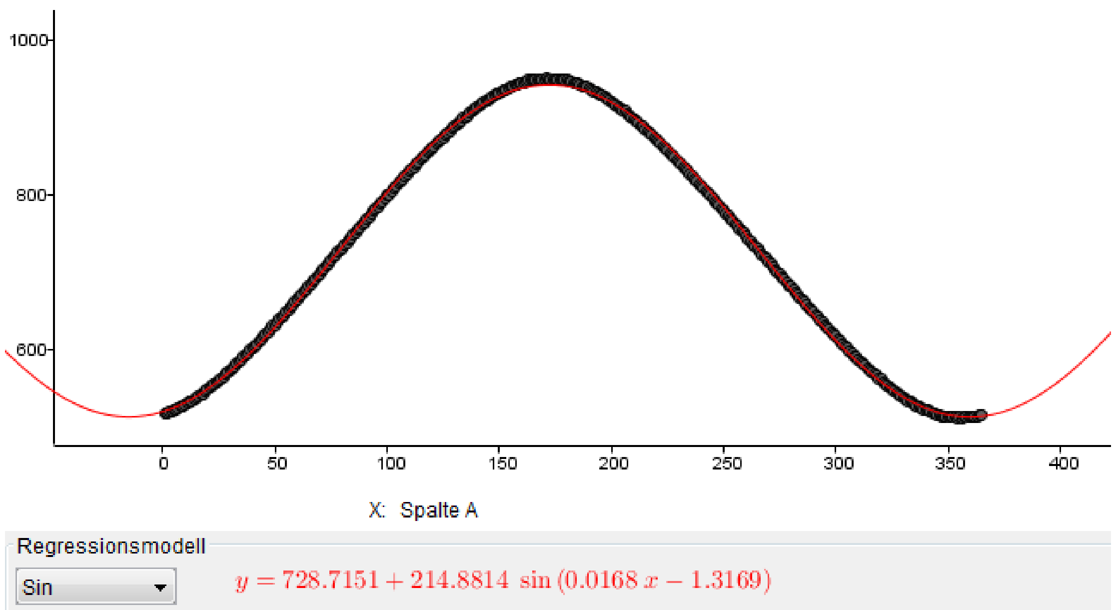

Sunrise
600
500
400
300
200
100
0
Sunrise [min]
0
50
100
150
200
250
300
350
400
Days


Sunset
1250
1200
1150
1100
1050
1000
950
900
Sunset [min]
0
50
100
150
200
250
300
350
400
Days


Length of Day
16,00
15,00
14,00
13,00
12,00
11,00
10,00
9,00
8,00
Length of Day [h]
0
50
100
150
200
250
300
350
400
Days

It is only possible in GeoGebra to approximate the discovered curve with the sine function as the necessary tools do not exist in Excel.




With the help of the approximated function $f(x) = a \cdot \sin(bx + c) + d$, the data can be analyzed:

Average length of day: $d = 728{,}7 \text{ min} = 12 \text{ h } 9 \text{ min}$

Difference longest – shortest day: $2a = 429{,}8 \text{ min} = 7 \text{ h } 10 \text{ min}$

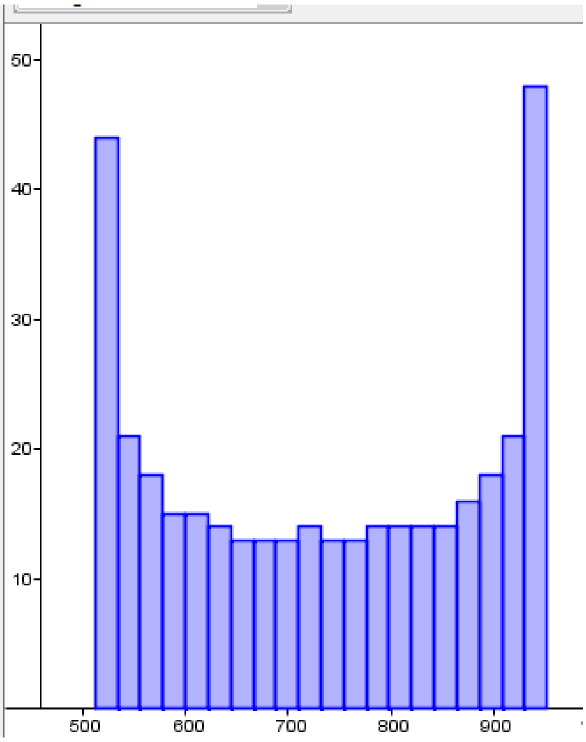
Period duration: $T = \frac{2\pi}{b} = 374$ Days (Problem: rounding errors)

Phase shift: $\varphi = \frac{c}{b} = 78{,}4$ Days

Thus, the 18th of December is the minimum or shortest day.

Comparison with actual results:

| | |
|---|---|
| n | 365 |
| Mittelwert | 734.2904 |
| σ | 150.0175 |
| s | 150.2234 |
| Σx | 268016 |
| Σx² | 205015996 |
| Min | 512 |
| Q1 | 589.5 |
| Median | 737 |
| Q3 | 879.5 |
| Max | 952 |




It is necessary to explicitly name the website on which the data should be collected as some cities have already put graphs on the length of day online. However, especially the analysis of data is difficult for the pupils.

## Performance Rating

Especially important are the completeness of the required report and the logic of the argumentation. In the next exam it is possible to ask a question about the project (e.g. which function describes the change of the length of day over the course of a year? Explain).

## Further Considerations

After the end of the project, the general sine function should be analyzed systematically.

a) Investigate the influence of the parameters $a$, $b$, $c$, and $d$ on the graph of the function
$$f(x) = a \cdot \sin(bx + c) + d.$$

Example: Take a close look at the graphs of $\sin(x)$, $\sin(2x)$, $\sin(3x)$, $\sin(0{,}5x)$, $\sin(0{,}2x)$.
Which changes does the parameter generate in the graph of the function?

b) Write a clear report about your observations.

c) Find adequate terms for the significance of the parameters.

# Identify the Lengths of Days over the Course of the Year in Your Home Town!

In the following, you can look at the change in the lengths of days over the course of the year in your home town. After that, you will be able to present your results graphically and approximate them with a suitable curve (regression model).

As a basis, you use data collected on the internet. It is your decision, which programs you would like to use in order to process and analyze the data.

## 1 Collect Data

Find information about the exact time of sunrise and sunset in your home town on this site:
http://aa.usno.navy.mil/data/docs/RS_OneYear.php

## 2 Analyse Data

Analyse the collected data:

- How long is an average day?
- By how many minutes differ the longest and the shortest day?
- Calculate the change in the lengths of days per day in minutes. Why does the change in some periods of time happen more rapidly than in others?
- How do the lengths of days change over the course of the year at the North Pole?
- If a day is longer at the North Pole than it is here, the temperature there should in comparison be higher! Argue.
- Are your data and results realistic?

## 3 Appendix

The sun is not punctiform, so why should the solar day be defined?

Does the sun really rise in the east? … Think of different places on earth.

## 4 Basis for Evaluation

Write a well-structured report about your observations and find logical terms for the different quantities.

# Which Shape is best for a Honeycomb Cell?

## Optimisation Tasks

Marion Zöggeler, Hubert Brugger, Karin Höller

### Intention

This learning environment is focused on the optimal shape of a honeycomb cell. This is an illustrative, interdisciplinary example of an optimization task. If the task is used in lower class levels, the focus is placed on the qualitative description of functions and the experimental approach at finding solutions. In higher class levels this learning environment can be used to work with differential equations of one or multiple variables.

A minimum of four lessons are recommended for this learning environment.

### Background of Subject Matter

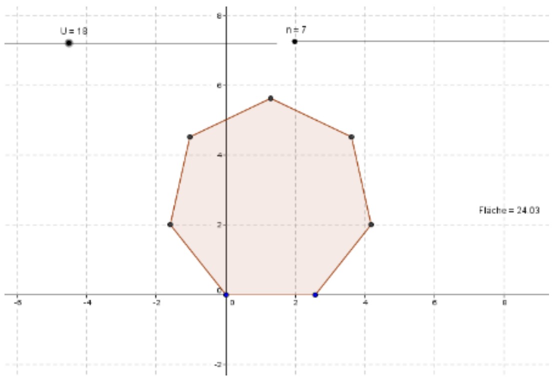
The tasks are supposed to guide the pupils to the optimal honeycomb cell. The first tasks are more simple tasks in the plane which have to be translated into the three dimensional space.

Here are a few suggested solutions:

#### For 1 Maximal Area

In GeoGebra, the different polygons with given perimeter are constructed and their areas are calculated. Here it can be seen that the area converges with an increase in vertices. The limit is the area of a circle.

The GeoGebra file can be downloaded at www.KeyCoMath.eu.



### Areas of Regular Polygons in a Circle

In Excel, a tabular overview of the areas of different polygons can be created.

Example of a circle radius of 10 cm:

| Number of vertices n | Central angle in degree measure | Central angle in radian measure | Area of the n-gon in $cm^2$ |
|---|---|---|---|
| 3 | 120,00 | 2,094395102 | 129,9038106 |
| 4 | 90,00 | 1,570796327 | 200,0000000 |
| 5 | 72,00 | 1,256637061 | 237,7641291 |
| 6 | 60,00 | 1,047197551 | 259,8076211 |
| 7 | 51,43 | 0,897597901 | 273,6410189 |
| 8 | 45,00 | 0,785398163 | 282,8427125 |
| 9 | 40,00 | 0,698131701 | 289,2544244 |
| 10 | 36,00 | 0,628318531 | 293,8926261 |
| 20 | 18,00 | 0,314159265 | 309,0169944 |
| 30 | 12,00 | 0,209439510 | 311,8675362 |
| 40 | 9,00 | 0,157079633 | 312,8689301 |
| 50 | 7,20 | 0,125663706 | 313,3330839 |
| 60 | 6,00 | 0,104719755 | 313,5853898 |
| 70 | 5,14 | 0,089759790 | 313,7375812 |
| 80 | 4,50 | 0,078539816 | 313,8363829 |
| 90 | 4,00 | 0,069813170 | 313,9041318 |
| 100 | 3,60 | 0,062831853 | 313,9525976 |
| 150 | 2,40 | 0,041887902 | 314,0674030 |
| 200 | 1,80 | 0,031415927 | 314,1075908 |
| 250 | 1,44 | 0,025132741 | 314,1261930 |
| 300 | 1,20 | 0,020943951 | 314,1362983 |
| 350 | 1,03 | 0,017951958 | 314,1423915 |
| 400 | 0,90 | 0,015707963 | 314,1463462 |
| 450 | 0,80 | 0,013962634 | 314,1490576 |
| 500 | 0,72 | 0,012566371 | 314,1509971 |

Result: The area of the polygon approximates the area of a circle.

The Excel file can be downloaded at www.KeyCoMath.eu.

For higher class levels, a calculation of the limit may be adequate:

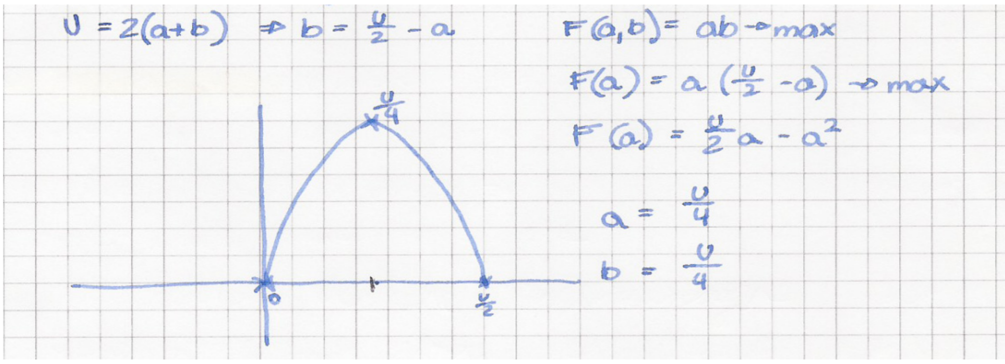
$$\lim_{n\to\infty} n \cdot \frac{r^2}{2} \cdot \sin\left(\frac{2\pi}{n}\right) \quad = \quad \lim_{x\to 0} 2\pi \cdot \frac{r^2}{2} \cdot \frac{sinx}{x} \quad = \quad r^2\pi$$

Here, the substitution $\frac{2\pi}{n} = x$ and the limit $\lim_{x\to 0} \frac{sinx}{x} = 1$ have been used.

## For 2 Special Case: Rectangle

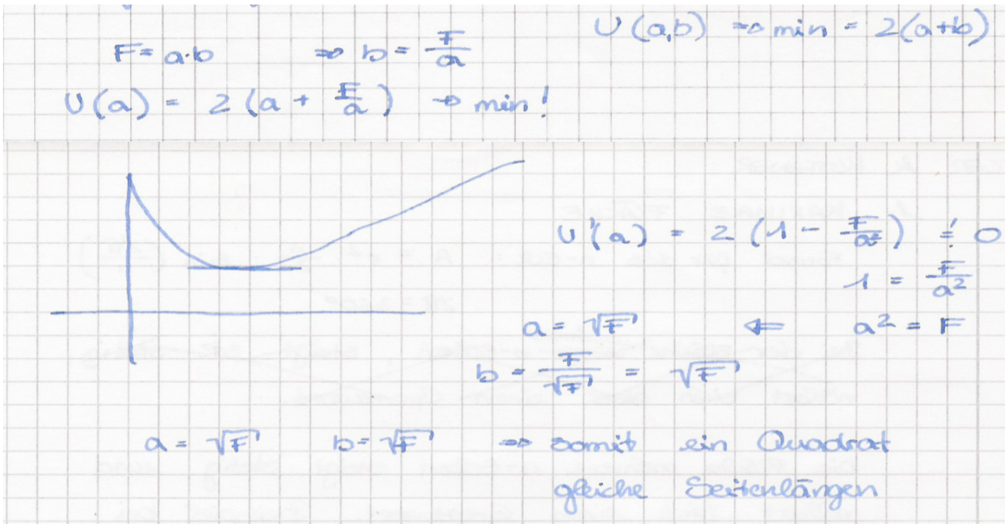
Here, a rectangle with given perimeter is to be considered: In which manner do the sides $a$ and $b$ have to be chosen in order to generate a maximal area?

This problem leads to the minimum of a quadratic function. A pupil's suggested solution:




## For 3 The Other Way Round: Minimal Perimeter

In this task the perimeter of a rectangle $P(a) = 2\left(a + \frac{A}{a}\right)$ with a given area $A$ should be examined in relation to a side length $a$. Again, a minimum ought to be found. In this case, a graphical representation can be helpful. In higher class levels, calculus will be used. A pupil's suggested solution:




## For 4 Special Case: Triangle

With Heron's formula $A = \sqrt{s(s-a)(s-b)(s-c)}$ , the area of a triangle with given perimeter can be expressed with two variables. Here $s = \frac{P}{2} = \frac{a+b+c}{2}$.

Experimental solutions can be found with Excel or GeoGebra. Exact solutions require the use of methods from mathematical analysis in two variables.

## For 5 From Plane to Space

In this task terms have to be translated from the plane to the three dimensional space.

- Perimeter equals surface area
- Area equals volume
- Rectangle equals cuboid
- Square equals cube

All pupils should be able to do this. Finding optimal solutions in three dimensional space, however, will be especially interesting for motivated pupils.

Example: Consider a cuboid with given volume and determine the sides *a*, *b* and *c* in the manner that the result is a surface area which is as small as possible.
A pupil's suggested solution:

5.2 $V = abc$ gegeben

$O(a,b,c) = 2ab + 2ac + 2bc \rightarrow min$

$c = \frac{V}{ab}$

$O = 2ab + 2\cancel{a}\left(\frac{V}{\cancel{a}b}\right) + 2\cancel{b}\left(\frac{V}{a\cancel{b}}\right) \rightarrow min$

$O'a = 2b - \frac{2V}{a^2} = 0$

$O'b = 2a - \frac{2V}{b^2} = 0$

$a = \frac{V}{b^2}$

$2b - \frac{2V}{\left(\frac{V}{b^2}\right)^2} = 0 \qquad 2b - 2\cancel{V} \cdot \frac{b^4}{V^{\cancel{2}}} = 0$

$2b - \frac{2b^4}{V} = 0 \quad | \cdot V$

$2bV - 2b^4 = 0 \quad | : 2$

$-b^4 + Vb = 0$

$-b^4 + Vb = 0 \quad | : b$

$-b^3 + V = 0$

$b^3 = V$

$b = \sqrt[3]{V}$

$a = \frac{V}{(\sqrt[3]{V})^2} \quad \Big| \cdot \frac{\sqrt[3]{V}}{\sqrt[3]{V}} \qquad \cancel{a = \frac{V}{(V^{\frac{1}{3}})^2} = \frac{V}{V^{\frac{2}{3}}} =}$

$$\frac{\cancel{V}\cdot\sqrt[3]{V}}{\cancel{V}} = a$$

$$a = \sqrt[3]{V}$$

$$c = \frac{V}{ab} = \frac{V}{\sqrt[3]{V}\cdot\sqrt[3]{V}} = \frac{\cancel{V}\cdot\sqrt[3]{V}}{\cancel{V}}$$

$$c = \sqrt[3]{V} \qquad O''a = + \frac{4V}{a^3} \to \min$$

### For 6 Optimal Shape of a Honeycomb Cell – Hexagonal Base

Pupils should show that the plane can only be completely covered by triangles, squares or regular hexagons. At each vertex of a polygon in the plane at least three other polygons meet. The sum of the neighbouring interior angles is 360°. For an interior angle of a regular $n$-gon, it is valid that:

$$\alpha = \frac{(n-2)\cdot 180°}{n}$$

This has to be a divisor of 360°, which is only valid for $n = 3$, 4 and 6.

In order to find the minimal perimeter of the polygon, results from task 1 can be used.

### For 7 Optimal Shape of a Honeycomb Cell – Optimal Inclination Angle

The pupils can build a model of a honeycomb cell out of paper. This increases the understanding of the situation.
This task can only be used in higher class levels as calculus is needed.

## Methodical Advice

It makes sense to solve the tasks together with a partner or in groups. After this, the suggested solutions can be discussed in class.

## Performance Rating

The performance rating can be based on the drawing up of the models, the work on the computer, the presentation of the results, and the way of working.

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# Optimization task:
# Which is the Best Shape for a Honeycomb Cell?

In nature, building styles have improved through evolution and are now almost optimal – for instance the building of a honeycomb cell. But what is optimal in this context?

Solve these tasks in order to answer the question:

## 1 Maximal Area

Given is a piece of rope of a certain length. How can you enclose an area which is as large as possible? Try different geometric shapes. Document your results.

Which is the most adequate shape? Give reasons. Use Excel or GeoGebra to support your claim.

*Hint*: Formula for the area of a regular polygon: $A = n \cdot \frac{r^2}{2} \cdot \sin\left(\frac{2\pi}{n}\right)$

## 2 Special Case: Rectangle

Choose a rectangle with given perimeter. Determine the side lengths that generate the largest area. Argue your suggested solution. Are there different approaches?

## 3 The Other Way Round: Minimal Perimeter

Consider a rectangle with given area. Which side lengths should it have to generate a minimal perimeter? Argue your suggested solution.

## 4 Special Case: Triangle

Examine any triangle with a given perimeter. How long should the side lengths be in order to create a maximal area? Argue your suggested solution by considering the area as a function with two variables.

*Hint*: Heron's formula for the area of triangles: $A = \sqrt{s(s-a)(s-b)(s-c)}$, with $s = \frac{P}{2} = \frac{a+b+c}{2}$

## 5 From Plane to Space

Translate tasks 2 and 3 to the three dimensional space. Find optimal solutions for this as well.

## 6 Optimal Shape of a Honeycomb Cell – Hexagonal Base

Firstly the problem is considered in a plane. The honeycomb cells completely cover the plane. Which polygons are adequate for this?

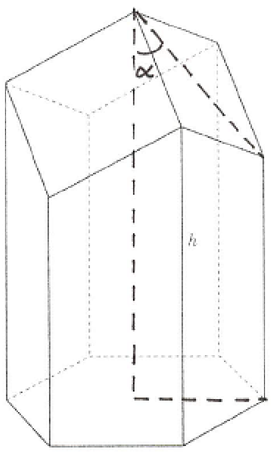
*Hint*: formula for the sum of the interior angles of a regular n-gon:

$$\alpha = \frac{(n-2)\cdot 180°}{n}$$

Which of these polygons has the minimal perimeter?

*Hint*: Use the solution of task 1.

## 7 Optimal Shape of a Honeycomb Cell – Optimal Inclination Angle




The figure shows the model of a honeycomb cell. You can see that the top is no plane hexagon. The biologist D'Arcy discovered that the surface area is just dependent of the inclination angle of the three areas forming the top of the cell. He even found a formula for this:

$$S(\alpha) = 6ab + \frac{3a^2}{2}\left(\frac{\sqrt{3}-\cos(\alpha)}{\sin(\alpha)}\right)$$

Here, $a$ is a side of the hexagon and $b$ is the longer side edge.

Determine angle $\alpha$ in the manner that it generates a minimal surface area.

# Functions and their Derivatives

## Courses of Functions, Monotony, and Curvature

Johann Rubatscher

### Intention

The aim of this worksheet is o make pupils familiar with the connection between a function and its derivatives. No calculations are needed in order to solve the tasks. Practical experience shows that about 50min are needed to complete the worksheet.

### Background of Subject Matter

Multiple tasks are aimed at an understanding of the relation between monotony and curvature.

### Methodical Advice

The pupils should solve the tasks alone or together with a partner. The tasks' level of difficulty is average.

### Performance Rating

In an exam, it is easily possible to test if the students have understood the relations and are able to argue their claims through varying the tasks.

# Functions and their Derivatives

May function $f$ be continuous and differentiable.

**Task 1**
May function $f$ be strictly monotonically decreasing in the interval I. What happens to functional value $f(x)$ in the interval if the $x$-value of the function is decreasing? Argue.

**Task 2**
In a right curve, may the function $f$ have the slope 4 in $x_0$. Which slope will $f$ have in $x_1$ in this right curve, if $x_1$ is located to the right of $x_0$? Argue.

**Task 3**
May $f''$ possess the value $-5$ in $x_0$. What does this tell you about $f'$ and $f'(x_0)$ in a neighbourhood of $x_0$ ? What can be said about $f(x_0)$? Argue.

**Task 4**
May $f'$ intersect the $x$-axis in $x_0$ from above. Which special point is $x_0$? Argue.

**Task 5**
If necessary, correct the following argumentation:

> In a right curve of a function $f$ the slope of the tangent decreases if $x$ moves from left to right. As the derivate of a function describes the slope of the tangent in a particular point, $f'$ and $f''$ are strictly monotonically decreasing in the right curve.

Argue if you have changed the argumentation.

**Task 6**
If possible, draw a sketch of a section of a function $f$ which has a left curve and is strictly monotonically decreasing. What can be said about $f''$ in this part of the curve? Argue.

**Task 7**
May function $f$ change from a right curve into a left curve in $x_0$. Additionally, may $f$ be strictly monotonically increasing close to $x_0$. Is this possible? Argue with the help of a sketch.

**Task 8**
May function $f$ change from a right curve into a left curve in $x_0$. Additionally, may $f$ be strictly monotonically decreasing close to $x_0$. Is this possible? Argue with the help of a sketch.

**Task 9**
May $f''$ be monotonically decreasing in an interval and possess an $x$-intercept in this interval. What can you conclude about $f$ in this interval? Argue.

**Task 10**
May $f'$ be strictly monotonically increasing in an interval I but may $f'$ possess only negative functional values in this interval. Sketch the course of $f$ in the interval and argue your solution.

# Important Lines in Triangles

## A Computer-Assisted Lesson

Walther Unterleitner, Manfred Piok, Maximilian Gartner

### Intention

The pupils construct the most important features of triangles with the help of GeoGebra and study their characteristics by changing the shape of the triangles. Four to six lessons are recommended for this tasks depending on the pupils‘ knowledge of GeoGebra.

### Background of Subject Matter

The most important points in triangles (Point of intersection of perpendicular bisectors as circumcentre, point of intersection of angle bisectors as incentre, centroid (centre of mass), and point of intersection of altitudes as orthocentre) are being constructed. The location of these points varies depending on the shape of the triangle; three out of the four are located on the Euler line. GeoGebra enables the pupils to test various cases and through this draw conclusions. Additionally, faster pupils can try to find the Fermat point.

### Methodical Advice

In order to work with this learning environment, pupils require knowledge of the following terms:

- Perpendicular bisector
- Angle bisector
- Median
- Altitude
- Distance

The first task can be used for demonstration purposes in order to explain GeoGebra’s tools. Pupils ought to work alone or together with a partner.

### Performance Rating

The pupils solve the tasks and write down their thoughts and results on the computer. The produced graphs from GeoGebra will be added to the document. This document will then be graded according to content, clarity, and the proficiency in construction and argumentation.

# The Most Important Lines in Triangles

For each task, create an own window in GeoGebra which you can access at any time.

## 1 Altitudes

Construct a random triangle and draw in the three altitudes. Construct their point of intersection O. Move the vertices of the triangle so that the orthocentre O is relocated as a result. Note down how the location of the orthocentre moves in relation to the shape of the triangle.

## 2 Perpendicular Bisectors

Construct the perpendicular bisectors of a triangle. Label their point of intersection as C.
Move the vertices of the triangle so that the point of intersection C is relocated as a result. Note down how the location of C changes in relation to the shape of the triangle.

## 3 Angle Bisectors

Construct the angle bisectors of a triangle. Label their point of intersection as I. Is it possible that I is located outside of the triangle? Argue.

## 4 Medians

Construct the line segments from the midpoints of the sides to the opposite vertices. Label their point of intersection as S and study its location.

## 5 Circles and Relations

Which one of the points has the same distance from all three vertices of the triangle?

Which one of the four points has the same distance from all three sides of the triangle?

Measure the distance between the vertices and the four points of intersection and the distance between the four points of intersection and the three sides of the triangle. What do you notice?

Additional Task: Is there a point for which the sum of the distances to the vertices is minimal?

## 6 Connection between the Different Points of Intersection

Construct a triangle that covers most of the screen. Draw the now familiar special lines and their points of intersection into the triangle. Fade out the special lines so that a clear picture remains. Move the triangle and track the motion of the four points of intersection. What do you notice?

Are you able to move the vertices in the manner that all four points of intersection are situated on a line?

# Measuring Distance and Height

## Functioning of Apps

Marion Zöggeler, Hubert Brugger, Karin Höller

### Intention

Smartphones and tablet computers offer a variety of apps which are often based on mathematical calculations. This is the case with the measuring of distance and height. The following task is supposed to encourage the pupils to question these technical gadgets.
For this topic approximately 1-2 lessons are recommended.

### Background of Subject Matter

In order to fulfill the task, basic knowledge in trigonometry is required.

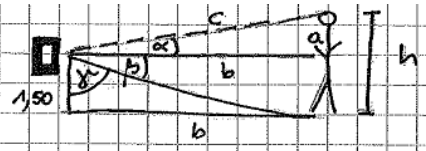
### Methodical Advice

This task is designed for groups consisting of two to three pupils. Great importance is attached to the documentation of the pupils' works and the description of their approaches. The following questions are supposed to steer this documentation.
For this experimental task, a sufficient amount of smartphones or tablet computers have to be at hand. Additionally, tapes for measuring, rulers, protractors, and cords should be within reach.

### Performance Rating

Each pupil's solution contains a sketch that shows which quantities are given or are being measured. The missing quantities are calculated through the application of trigonometric relations. The performance rating takes these two aspects into consideration. The process of measuring itself or the search for additional information (e.g. in the app's user manual) should also contribute to the final rating.

### Pupil's Example




$\gamma = 90 - \beta \quad \tan(\gamma) = \frac{b}{1,50} \Rightarrow b = \tan(\gamma) \cdot 1,50$

$\tan(\alpha) = \frac{a}{b} \Rightarrow \alpha = \tan(\alpha) \cdot b$

$h = 1,50 + a$

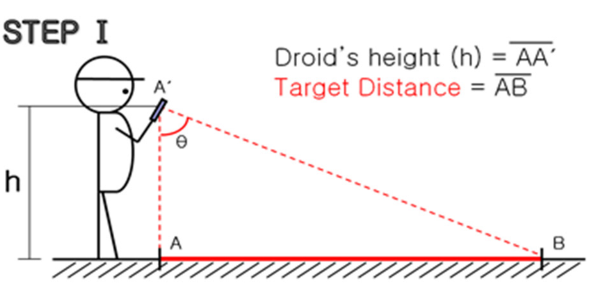
# Measuring Distance and Height

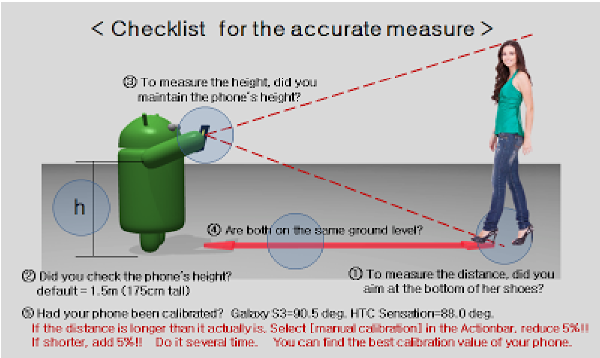
## Functioning of Apps

Smartphones and tablet computers offer a variety of apps which make use of mathematical calculations. An example is the measuring of distance and height.

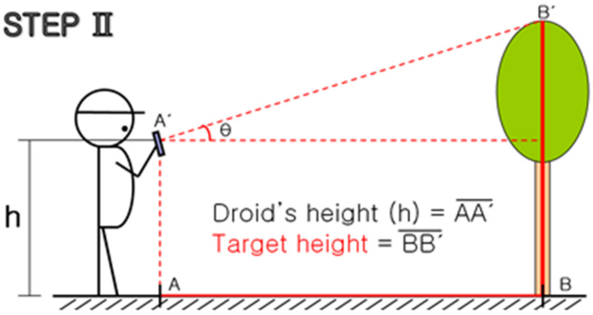
### Small Distances (1 - 50 m)

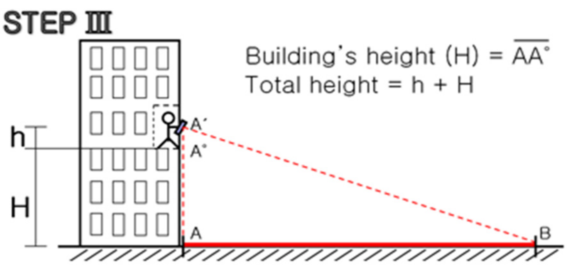
Taken from the “Smart Measure Lite“-Manual:




- The measured length is for reference.
- Before using this App, calibrate your devices with known distances.
- If the measured distance is **longer** than it, **reduce** by 5% at manual calibration. If **shorter**, **increase** by 5%. Do it several times. You can find your own best calibration.




© Smart Tool co

## Tasks

Shown above are the instructions for the app “Smart Measure“.

Measure some distances and heights with your smartphone or tablet PC and check the results with a tape measure. Compare.

Now we want to take a closer look at the app:

- Which quantities are necessary for the app?
- The device can measure angles of inclination. Which angles of inclination are necessary for calculating the distance and height of an object?
- By which means are distance and height calculated?
- Draw a sketch, measure the required quantities with a tape measure and calculate the distance and height yourself using trigonometry.

# Thinking in Functions

## Introduction to the Topic of Functions

Dorothea Huber

### Intention

In this learning environment, pupils should work extensively with functional relations and their graphs.
According to Kröpfl (2007), the two main ideas of functions are the dependence of a variable of one or more variables and the course of the function. The course of the function is expressed in the changes which are noticeable in the value table and the graph of the function. In daily life, rather few functions are fundamental types of functions. The pupils are provided with a general introduction to the field of functions. At this point they will not be acquainted with linear and quadratic functions as special types of functions.

### Background of Subject Matter

In this learning environment, the pupils become acquainted with the different types of functions, especially those which cannot be described in a functional equation. They learn that some graphs do not portray a function. Additionally, they hear about continuous and discrete functions. Furthermore, they repeat percentage calculation and the formula for the area of a rectangle and study graphs of functions according to their rate of change.

The following thematic aspects will be considered:

- Time-Distance-Functions (linear and quadratic functions)
- Functions describing the process of filling a container (Volume → Water level)
- Power functions
- Exponential functions

### Methodical Advice

In order to guide and help the pupils more easily (e.g. with the ultrasound scanner), most of the lesson is organised according the market place method. Here it is very important to assist especially weaker pupils. A possible course of the lessons could be:

#### 1st lesson

At first, a personal graph of the pupils‘ well-being on the day before is drawn by the pupils themselves. A few interesting graphs are presented in class.
Then the class is separated. One part is working in groups of two on stall 2 “walking along graphs”. The other half of the class is working with the ultrasound scanner. This scanner measures the distance of an object in relation to the time. The teacher assists the students with the first measurements. This stall consists of tasks of varying difficulty and the solutions can be checked by the pupils themselves.

At stall 2 “walking along graphs“, students walk along Time-Distance-Graphs in groups of two. There is also an impossible graph. The pupils should recognize that this cannot be the graph of a function. The homework deals with the “hoisting of a flag”.

### 2nd lesson

The pupils continue the work in the market place. At stall 3 they measure the water level of an Erlenmeyer flask in relation to the volume. At the end of this task, the pupils should be able to match graph and container. Stall 4 “rectangles” deals with indirect proportional functions. The formula for the area of a rectangle is repeated. Stall 5 is an example for a discrete exponential function.

### 3rd lesson

The work in the market place is completed. The ideas and results of individual stalls can be discussed in class if necessary.

## Practical Experience

The following observations have been made in class:

- Instead of the expected argument “it is not possible to be in two places at once“, “time cannot stand still“ is used to argue the impossible graph.
- Especially the course of functions is regarded in this approach.
- Regularity in the loss of drawing pins is not recognized.

## Performance Rating

The acquired content, knowledge, and skills can be tested in an exam in the following manner:

- Describing graphs of movement
- Identifying/ describing the water level of a graph
- Similar functional relationships: e.g. time → speed, time → pages read
- Experiment: folding a sheet of paper repeatedly in the middle. Recognizing the number of paper layers in relation to the number of foldings as a functional relation and being able to depict this.

## Bibiliography

Büchter A., Henn H.-W. (Hrsg.) (2008): Der Mathe-Koffer, Friedrich Verlag

Kröpfl, B. (2007): Höhere mathematische Allgemeinbildung am Beispiel von Funktionen, Profil Verlag, München Wien

Mathematik 5 bis 10, Heft 8|2009, Friedrich-Verlag

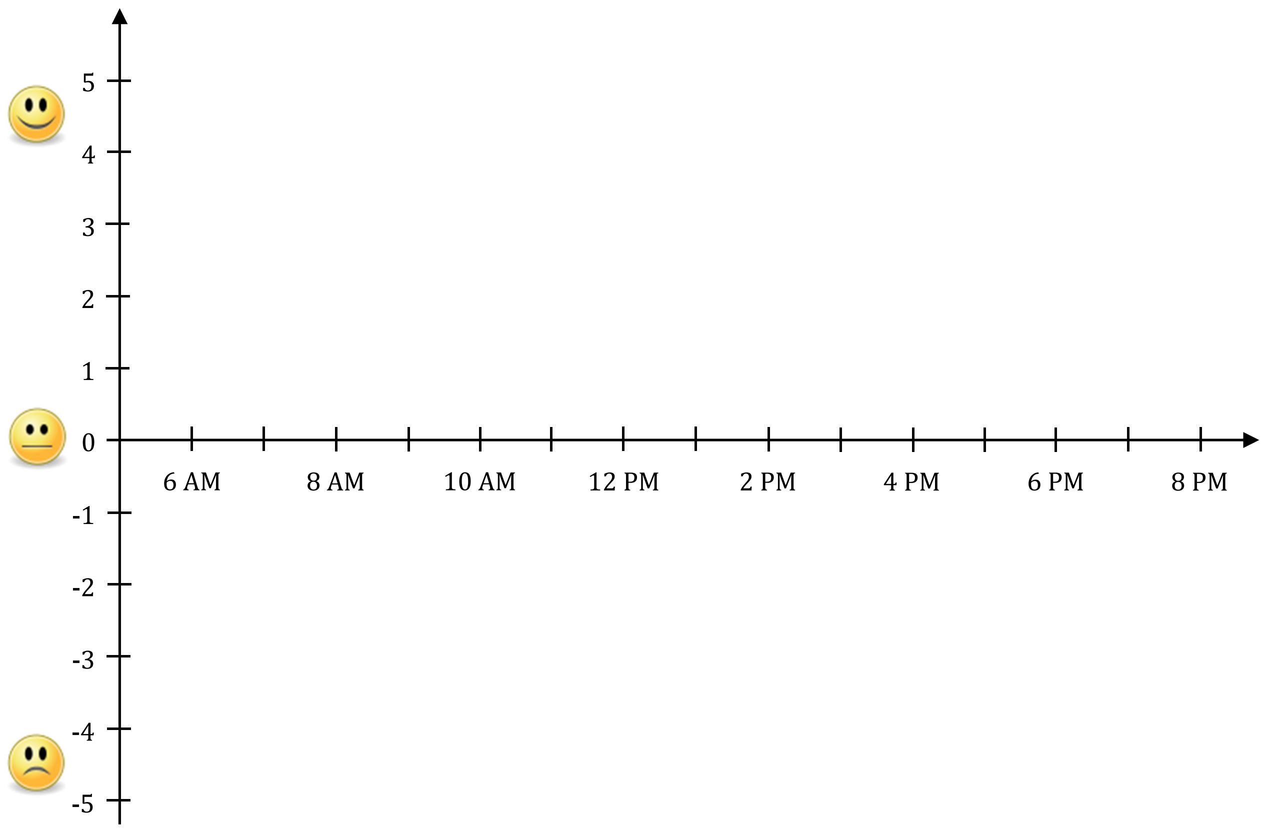
http://www.nawi-aktiv.de/umaterial/labor/laborgeraetequiz_zuordnung.htm

http://www.individualisierung.org/_neu/download/funktionen_pool1.pdf

# Introduction to Thinking in Functions

## Personal graph of my well-being

Draw yesterday's graph of your well-being.




(see: Mathematik 5 bis 10, Heft 8|2009, Friedrich-Verlag)

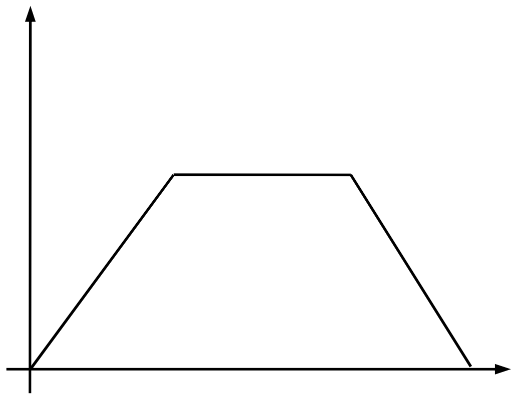
# Working in the Market Place

## 1 Market Stall “Ultrasound Scanner“

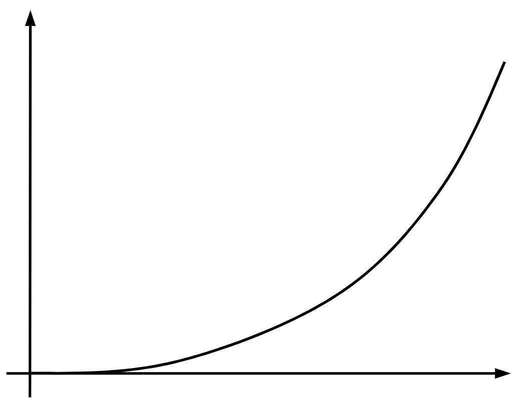
Move your hand away from the ultrasound scanner and draw the graph.

Move your hand closer to the ultrasound scanner. Draw the graph and think about which quantity should be measured on the x-axis and which quantity on the y-axis.

In which way do you have to move your hand in order to generate the following graph? Try.

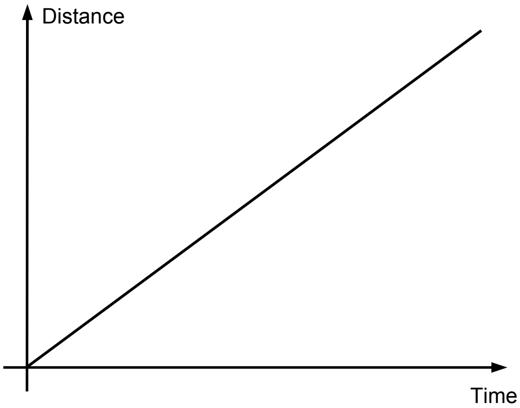


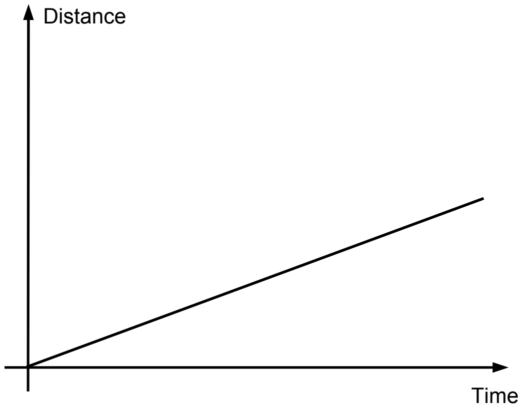
In which way do you have to move your hand in order to generate the following graph? Try.

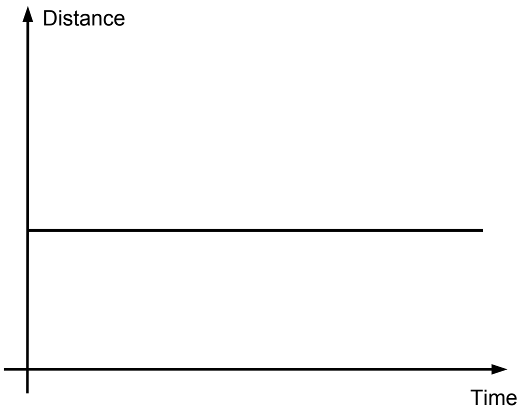


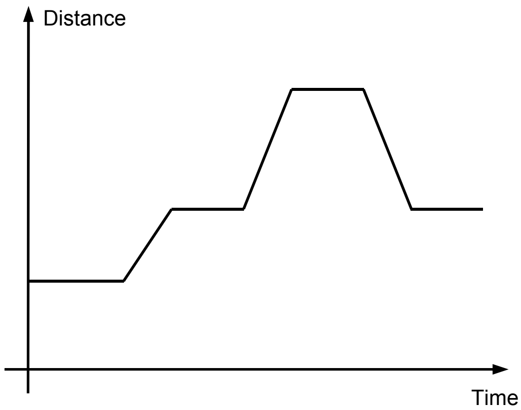
## 2 Market Stall “Walking along Graphs“

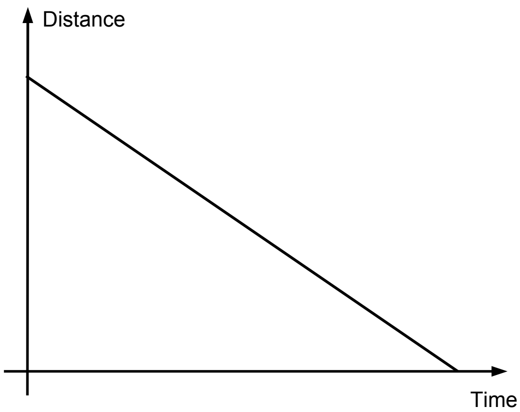
Choose a wall and walk along the graphs. The graphs should describe your distance to the wall in relation to the time. Additionally, you should describe your movements in words. If it is not possible to put some graphs into practice, give reasons for this.

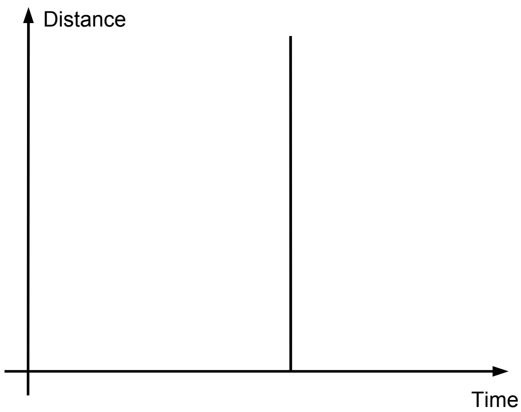


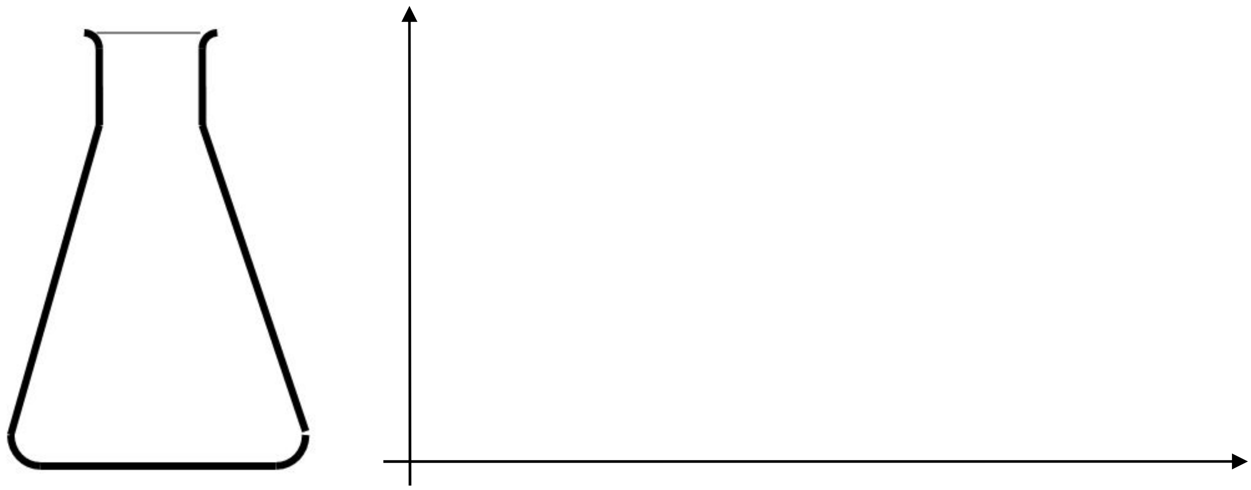




(see: Mathematik 5 bis 10, Heft 8|2009, Friedrich-Verlag)

## 3 Market Stall “Filling Containers“

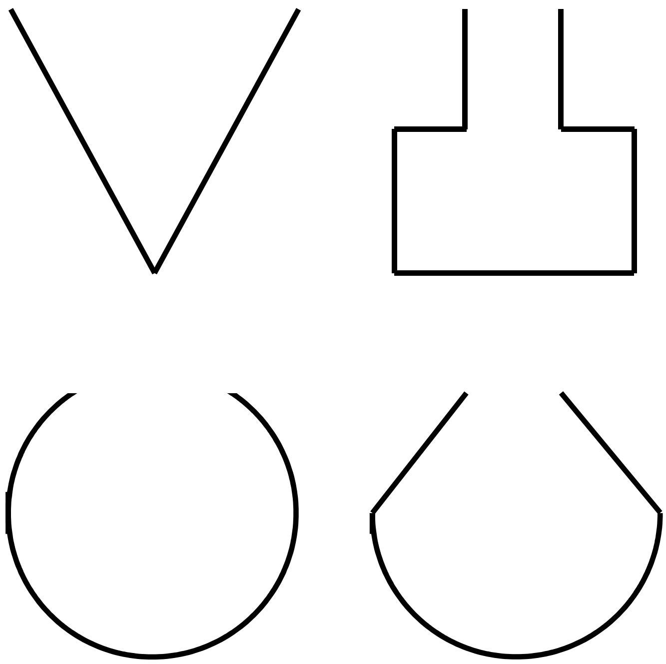
For this experiment, an Erlenmeyer flask, a ruler, and a graduated cylinder are necessary.

Check how the water level changes when the Erlenmeyer flask is evenly filled with water. Represent your findings graphically. Label the x-axis volume and the y-axis water level.



(see: http://www.nawi-aktiv.de/umaterial/labor/laborgeraetequiz_zuordnung.htm)

Find the graph which describes the process of filling the containers.



(see: Mathematik 5 bis 10, Heft 8|2009, Friedrich Verlag)

Describe the relation between container and graph. Find examples of your own.

## 4 Market Stall “Rectangles“

Take a sheet of paper and cut out four different rectangles with areas of 9 cm² !

Draw a coordinate system in which you can compare the connection between the side lengths of possible rectangles. The axes should be labelled with the two side lengths a and b of the rectangle. (Scale: 1 square ≙ 1 cm)

With which formula can you calculate the second side length if one side is given?

Glue the rectangles on the chart in an appropriate manner.

Does the graph intersect the axes? Give reasons.

If you have completed stall 5 already: Which differences and similarities can you discover between the two graphs?

## 5 Market Stall “Loss of Drawing Pins“

When you let drawing pins fall to the floor, some will land on their backs while others will land on their sides. Now your task is to investigate according to which rule this happens. You will need 100 drawing pins and a paper cup for throwing the pins.

Throw the drawing pins to the floor and sort out the ones lying on their sides. Note down the number of the rest of the pins. Repeat the experiment with the remaining pins four times.

Represent your findings graphically. Label the x-axis with “number of throws” and the y-axis with “number of remaining drawing pins“.

Are there regularities? Which ones?

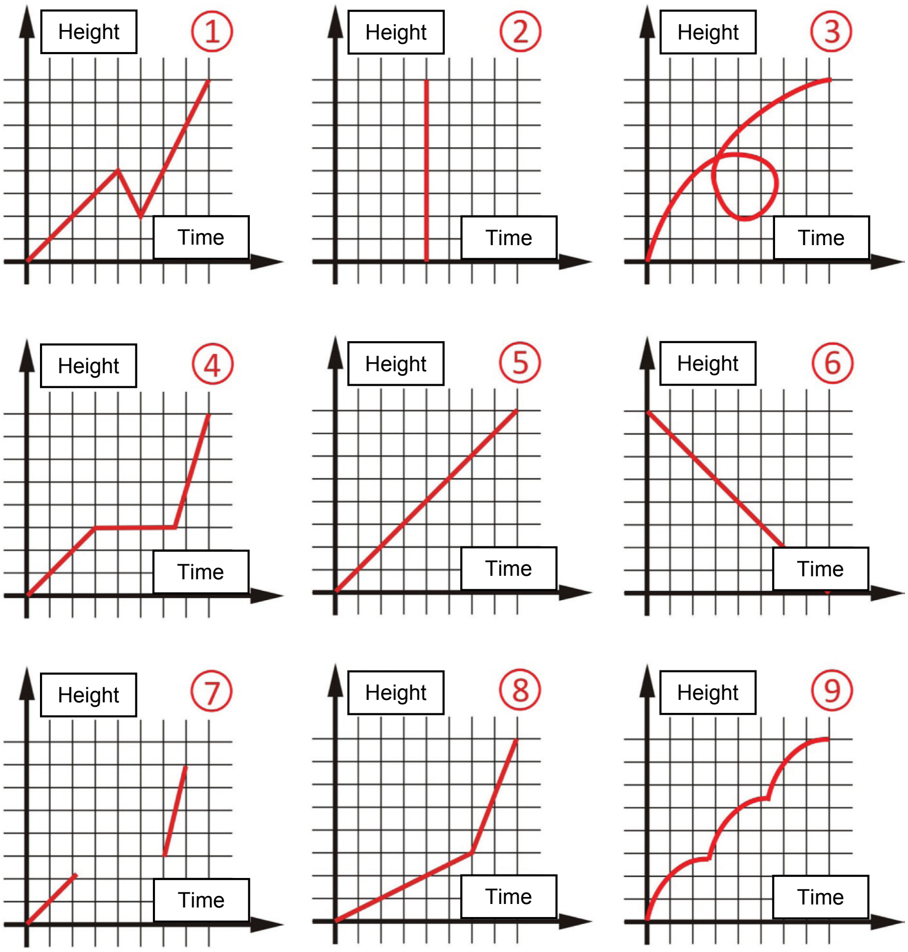
If you have completed stall 4 already: Which differences and similarities can you discover between the graphs?

(see: Mathekoffer, Friedrich-Verlag)

# Homework

## 1 Hoisting a Flag

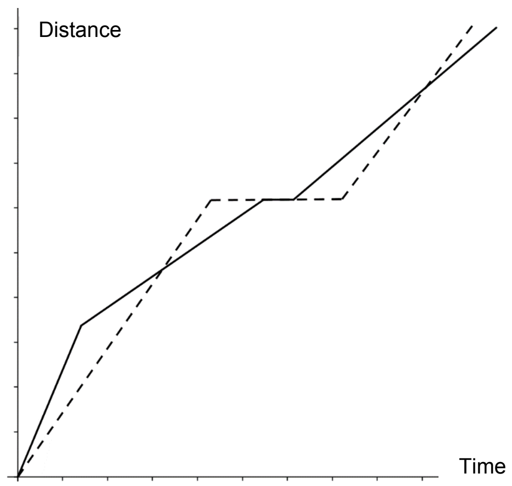
The nine figures partially describe the hoisting of a flag. You may have noticed that this process can be done quite differently. Some people quickly hoist the flag, others take small breaks, etc. Describe, if possible, how the hoisting of the flag happens in each figure.




(see: http://www.individualisierung.org/_neu/download/funktionen_pool1.pdf)

## 2 Bikerace

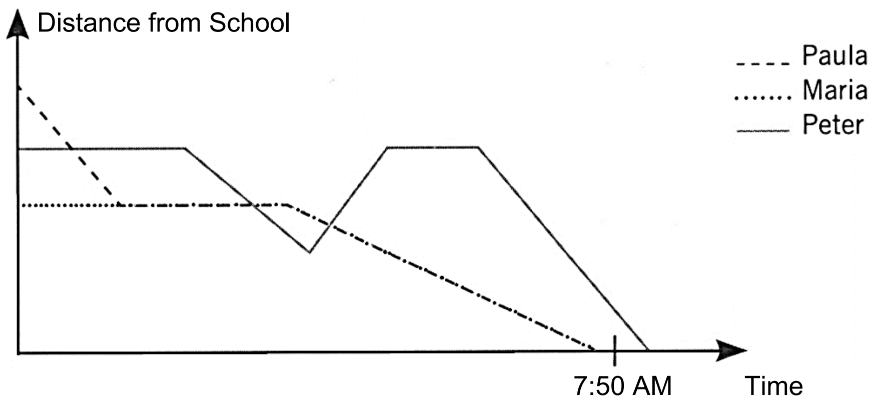
The diagram describes a bike race.
Comment on the race like a radio reporter.



# Test about Functions

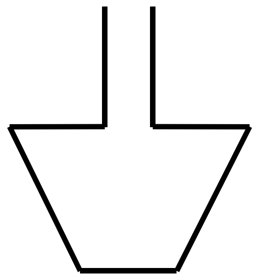
## Task 1

Peter, Paula und Maria are classmates and live on the same street. Their school is located at the end of the street. Every morning they walk to school, which starts at 7:50 AM. The graph shows where they were yesterday morning at different times. What happened? (It does not have to be an exciting story but it has to be clear that you are able to read graphs.)




## Task 2

Draw the graph which describes the process of filling the container.



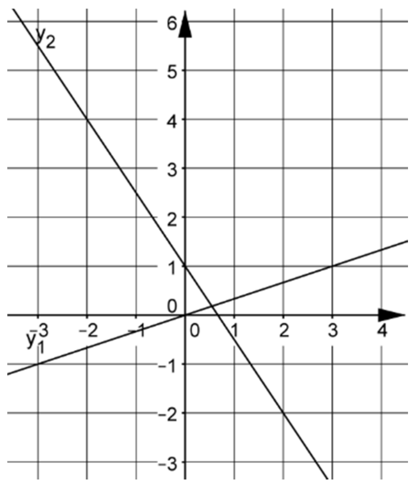
## Task 3

What do the graphs of linear functions look like? What can be seen in their value tables?

Draw the graph of: $y = \frac{1}{2}x + 3$

## Task 4

Determine the functional equations of the graphs.



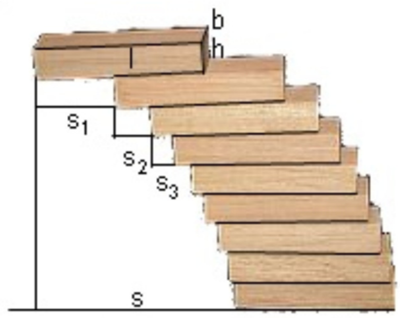
# Wood Stack – Where is the Limit?

## Discovering the Harmonic Series

Aron Brunner, Johann Rubatscher

### Intention

The question on how far equal wooden cuboids (e.g. dominoes, CD-covers) with the dimensions $l$, $b$, and $h$ in a stack can be moved without the stack collapsing has been asked for centuries. The maximal overhang $s$ of the cuboid on top should be discovered here. New developments can be found in the bibliography.




https://de.wikipedia.org/wiki/Harmonische_Reihe

### Background of Subject Matter

A stack of cuboids does not collapse if the foot of the common centre of mass of the cuboids does not leave the cuboid below. The maximal shift distance $s$ can be discovered by finding the centre of mass. Through adding new cuboids, $s$ can be described through the series

$$s \;=\; l/2 + l/4 + l/6 + l/8 + l/10 + l/12 + l/14 + \cdots,$$

which is equal to

$$s \;=\; \frac{l}{2} \cdot \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \cdots\right).$$

The brackets contain the harmonic series, which does not converge. Thus, by adding cuboids any shift distance can be reached.

### Working out the Results

The first cuboid on top can be shifted by $s_1 = l/2$ as then its centre of mass rests over the edge of the cuboid below.

The centre of mass of two cuboids is located in the middle of the centres of mass of the two cuboids on their own (symmetry). Thus, the two cuboids together can be shifted by $s_2 = l/4$.

The centre of mass of three cuboids, however, is not located in the middle of the centre of mass of the two cuboids and the third but is shifted in the direction of the centre of mass of the two cuboids as the two possess more mass. Due to this, the centre of mass is shifted in the ratio 2:1, which is equivalent to a third of $l/2$. Thus, it is valid that:

$$s_3 = 1/3 \cdot l/2 = l/6$$

The centre of mass of four cuboids is shifted according to the ratio 3:1, which leads to:

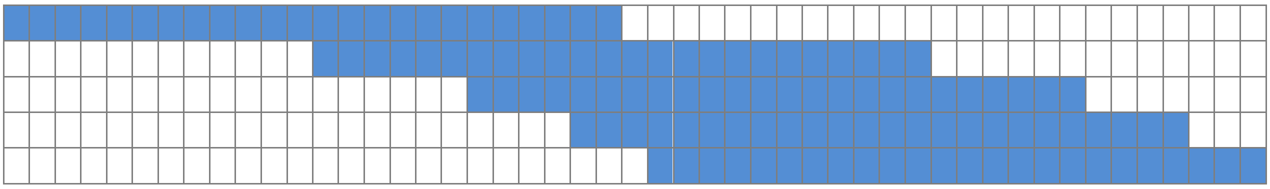
$$s_4 = 1/4 \cdot l/2 = l/8$$

The centre of mass of five cuboids is shifted according to the ratio 4:1, which leads to:

$$s_5 = 1/5 \cdot l/2 = l/10$$

Continuation for $s_5, \; s_6, \; s_7, \ldots$ generates:

$$s \;=\; s_1 + s_2 + s_3 + s_4 + s_5 + s_6 + \cdots \;=\; l/2 + l/4 + l/6 + l/8 + l/10 + l/12 + \cdots$$



The continuation of the harmonic series should be pointed out to the pupils. A spreadsheet program can calculate the harmonic series up to about $n = 200.000$.

## Bibliography

Pöppe, C. (2010): Türme aus Bauklötzen – Mathematische Unterhaltungen, Spektrum der Wissenschaft, September 2010, S. 64

Paterson, M., Peres, Y., Thorup, M., Winkler, P., Zwick, U. (2008): Maximum Overhang, arXiv.org, Cornell University Library, http://arxiv.org/pdf/0707.0093v1.pdf

# Wood Stack – Where is the Limit?

https://de.wikipedia.org/wiki/Harmonische_Reihe

## 1 Description of the Task and First Experiments

Take two equal wooden cuboids (dominoes or CD-covers) of length $l$, width $b$, and height $h$ (with $h$ a lot smaller than $l$ and $b$), stack them on top of each other and move the cuboid on top so far in longitudinal direction that it sticks out as far as possible. Put a third equal cuboid underneath the two first ones and move the two upper cuboids again as far as possible. Continue this process.

## 2 Calculation

Consider with pen, paper, pocket calculator, etc., how far you can move the cuboids successively if the total number of blocks is $n = 2$, $n = 3$, $n = 4$, $n = 5, \ldots$?
How far is the total shift distance $s$ in each case?

## 3 How far?

Which is the number $n_{max}$ of wooden blocks, with which a stack of this manner can be built that does not collapse?

## 4 With Help from the Computer

Calculate $n_{max}$ with a spreadsheet program.

## 5 Further Considerations

How big is $s$, if each cuboid is moved by l/3 or l/4?

What would happen with wooden disks?

What would change if you were allowed to use counterweights or any other additions?

Read the following article:

Paterson, M., Peres, Y., Thorup, M., Winkler, P., Zwick, U. (2008): Maximum Overhang, arXiv.org, Cornell University Library, http://arxiv.org/pdf/0707.0093v1.pdf

# Moving to a New House

## Reflecting on Problem Solving

Panayiota Irakleous

## Goals (Mathematical Competence)

The students should acquire the following competences:

**Numbers**

- Use natural numbers until 1000 and decimals to solve problems.
- Compare and order natural numbers until 10000.
- Use the concept of ratio and solve proportional problems.
- Judge the reasonableness of calculated results.

**Algebra**

- Use verbal and algebraic expressions to represent additive and multiplicative relations.
- Understand the concept of variable, interpret and explain relations between variables.

**Geometry**

- Describe the position of objects, using concepts such as up-down, behind-in front of, next to, between, right-left.

**Measurement**

- Convert units within the metric system.
- Use standard units of measurement for length.
- Make estimations of distances.
- Interpret scale drawings.

**Statistics – Probabilities**

- Interpret and design frequency charts (bar chart, pie chart, linear graph, plots and tables).
- Perform the procedures of data classification, quantification, weighting and grouping.

## Key Competences for Lifelong Learning

- Digital Competence
- Social Competences
- Communication in the mother tongue
- Learning to Learn
- Sense of Initiative

# Moving to a New House

## A Warm up activity

*Family stories …*

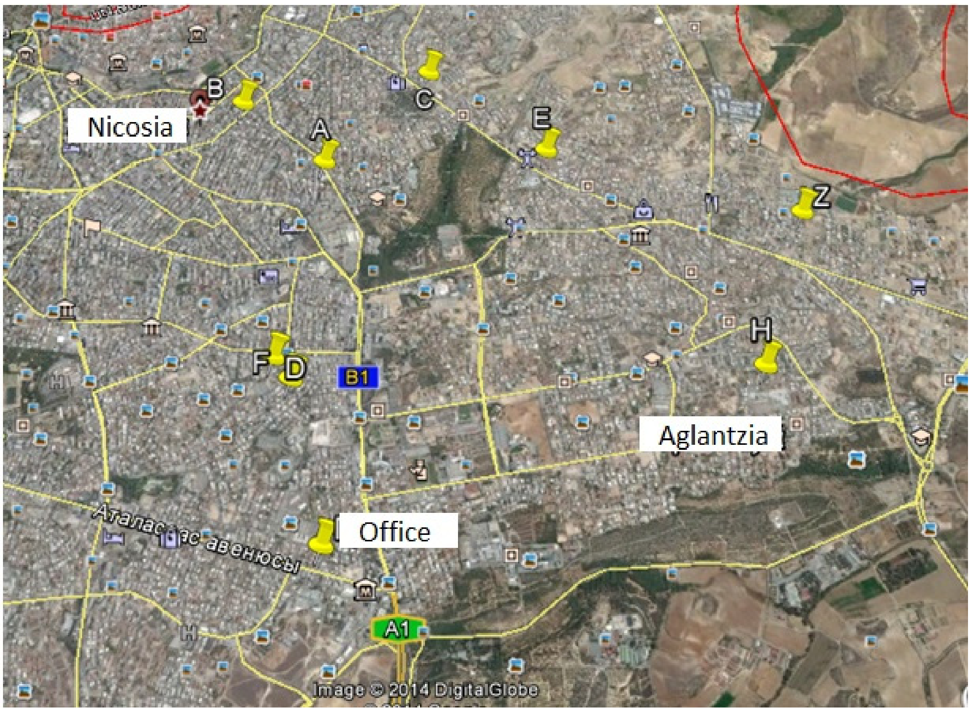
Leonidas and Ioanna have rented a furnished apartment at the center of Nicosia. They pay 620 euros per month in rent. They have two children, George and Marilia, who are twins. They are 7 years old and go to the A' Primary School in Latsia.

Both Leonidas and Ioanna have worked as accountants in the same office for 8 years, but last month Ioanna was fired. Thus, they want to move to a new house. The two parents believe that their children should still go to the same school, in order to protect their psychological balance. They seek to find an affordable apartment that meets their basic needs. To this end, they have conducted a market research and have collected information about two-bedroom apartments, as shown in the table below.

Apartments' features

| | Apartment | Area_in_square_meters | Year_of_construction | Parking_space | Storage | Furniture | Rent | Other_expenses | Floor |
|---|---|---|---|---|---|---|---|---|---|
| 1 | A | 65 | 1990-1995 | Covered | Yes | Full | 500 | 20 | 1st |
| 2 | B | 89 | 1990 | Not available | Yes | Two beds | 450 | 20 | Ground floor |
| 3 | C | 110 | 2000 | Covered | No | Full | 450 | 40 | 3rd |
| 4 | D | 75 | 2005 | Covered | No | Living room | 400 | 20 | 2nd |
| 5 | E | 85 | 1985-1990 | Non-covered | Yes | Unfurnished | 400 | 20 | 1st |
| 6 | Z | 70 | 1985 | Non-covered | Yes | Full | 420 | 20 | 4th (penthouse ) |
| 7 | H | 90 | 2003 | Non-covered | No | Living room | 510 | 0 | Ground floor |
| 8 | U | 75 | 1996 | Covered | Yes | Full | 470 | 0 | 2nd |
| 9 | I | 98 | 2007 | Covered | Yes | Unfurnished | 400 | 30 | 1st |
| 10 | K | 85 | 1994 | Covered | Yes | Three beds | 450 | 0 | 3rd |
| 11 | L | 120 | 2002 | Not available | No | Full | 520 | 30 | 4th (penthouse ) |
| 12 | M | 80 | 2009 | Not available | Yes | Full | 470 | 32 | Ground floor |
| 13 | N | 100 | 2010 | Non-covered | No | Unfurnished | 460 | 0 | 3rd |

Below you can find a map that shows the position of each apartment as well as the office where Leonidas works.



*Comprehension questions*

1. What problem does the family face?

2. How far is the apartment B from the office?

3. Which apartment is closest to Leonidas' office?

4. Which apartment has the highest rent? Which apartment has the highest rent in relation to its area? Are these apartments the same? Why?

## B Problem

You should classify the apartments into three groups: very suitable for the family, moderately suitable and unsuitable. Write a letter to describe the method you applied in order to classify the apartments. The parents are going to use this method in the case of a future moving to a new house.

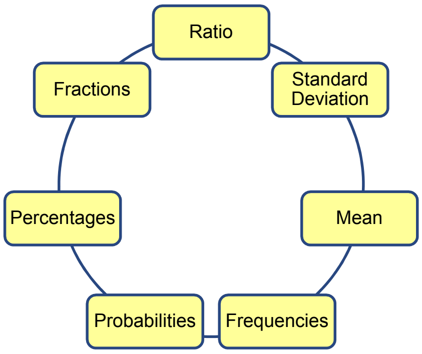
## C Reflection

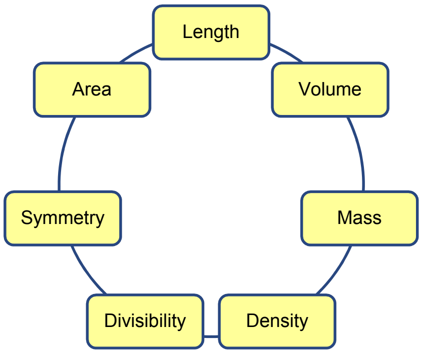
1. Step Diagram

Draw a step diagram to represent "changes in thinking" that your group went through during the solution of the problem as well as your level of engagement.

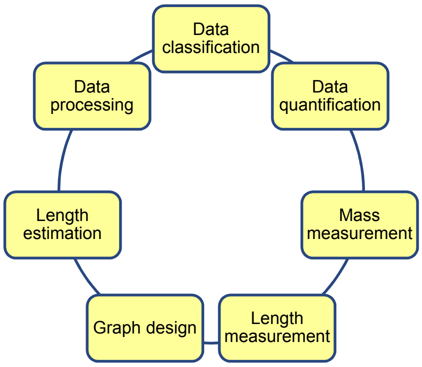
Start of session End of session

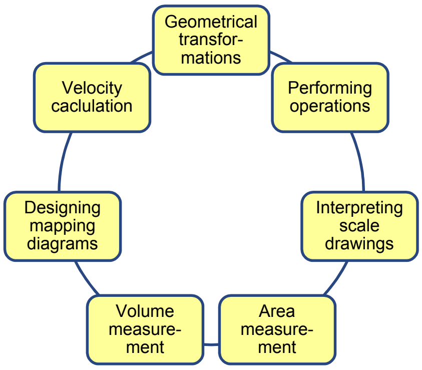
2. Which mathematical concepts did you use during the solution of the problem?




3. Which mathematical procedures did you use during the solution of the problem?




4. In your opinion, how well did you understand the concepts you have used? Explain why.
   - Not at all
   - To a small extent
   - To a moderate extent
   - To a large extent
   - Absolutely

5. How difficult was the problem for you? Explain why.
   - Very easy
   - Easy
   - Neutral
   - Difficult
   - Very difficult

6. In your opinion, which of the methods you have thought so far is the most appropriate?

# At the Zoo

## Planning a Route

Eleni Constantinou

### Goals (Mathematical Competence)

The students should acquire the following competences:

**Numbers**
- Use natural numbers and decimals to solve problems.
- Perform addition and subtraction of natural numbers.
- Use the concept of ratio and solve proportional problems.
- Judge the reasonableness of calculated results.

**Geometry**
- Describe the position of objects, using concepts such as up-down, behind-in front of, next to, between, right-left.

**Measurement**
- Use standard units of measurement for length.
- Convert units within the metric system.
- Interpret scale drawings.
- Recognize relations between units of time.
- Estimate time duration.

**Statistics – Probabilities**
- Organize and present data in tables.
- Perform the procedures of data classification, quantification, weighting and grouping.

### Key Competences for Lifelong Learning

- Digital Competence
- Social Competences
- Communication in the mother tongue
- Learning to Learn
- Sense of Initiative

### Tools available

The students should have the following tools for their work:
- Map of the zoo (digital and hard copy)
- Family preferences
- Advice of friends who have visited the zoo
- PC
- Stopwatch
- Measuring tape
- Skitch Touch application

# At the Zoo

## Problem


Erodotos and his family are in Barcelona for holidays. One of Barcelona's top touristic attractions is the Zoo. They are planning to visit the zoo on Friday (26/03), but they will have only 2 ½ hours available.

You should help the family to plan the most appropriate route in order to see as many animals as possible and also to decide the best time to visit the zoo. Additionally, you have to present this route, explaining the reasons why the particular route is appropriate for the family and for any other visitor who has less than 3 hours to visit the zoo.


## Family preferences

The family members have special interests:
*"I would like to see …*

*… the lions and the gorillas."* (Mr. Nikos)
*… the seals and the flamingos."* (Mr. Danae)
*… the elephants and the kangaroos."* (Aphrodite, 6 years old)
*… the dragons, the crocodiles and the camels."* (Achilleas, 10 years old)

## Advice of friends who have visited the zoo

*"The dolphins' show is incredible! You should definitely watch it!"* (Georgia)
*"The lions sleep from 12 o'clock until 5 o' clock, so it is difficult to see them."* (Alexander)

# The Energy Problem

## Working in Complex Real-Life Situations

Panayiota Michael, Elena Sazeidou, Stella Shiakka

## Goals (Mathematical Competence)

The students should acquire the following competences:

**Numbers**

- Use natural numbers until 10000 and decimal numbers to solve problems.
- Perform addition, subtraction, multiplication and division of natural numbers and decimals.
- Compare and order natural numbers until 10000.
- Use the concept of ratio and solve proportional problems.
- Judge the reasonableness of calculated results.

**Statistics – Probabilities**

- Describe and compare datasets by using the concept of arithmetic mean.
- Perform the procedures of data classification, quantification, weighting and grouping.

## Key Competences for Lifelong Learning

- Social Competences
- Communication in the mother tongue
- Learning to Learn
- Sense of Initiative

# The Energy Problem

## A Problem

The University of Cyprus senate has decided to give the opportunity to a group of students to choose the most appropriate type of lamp for a specific lecture hall. The dimensions of the particular lecture hall are 24 m x 20 m. The maximum amount of money that can be spent on the lighting of this hall for 12.5 years is € 800.

## B Problem

You are a member of the Central Committee of Electricity Authority of Cyprus. It is your duty to describe a procedure by which the customers will be able to choose the lamp that meets their needs, based on the following table.

## C Data

| Lamp | Electric power (Watt) | Luminosity (Lm) | Lm/Watt | Number of days in use* | Number of replacements over 25 years | Cost per lamp (EUR) | Electric power (kW) for 25 years | Cost (kWh) for 25 years |
|---|---|---|---|---|---|---|---|---|
| *Led* | 7 | 256 | 36,5 | 9000 | 1 | 35 | 252 | 0,07 |
| *Halogen* | 17 | 256 | 15,07 | 500 | 19 | 2,50 | 612 | 0,07 |
| *High-pressure sodium* | 273 | 256 | 0,93 | 5000 | 2 | 14,96 | 9.828 | 0,07 |
| *Light bulbs* | 24,6 | 256 | 10,4 | 250 | 25 | 0,75 | 885,6 | 0,07 |
| *Fluorescent* | 95,7 | 256 | 2,67 | 2.500 | 4 | 1,30 | 3.445 | 0,07 |
| *Mercury* | 347,2 | 256 | 0,73 | 4000 | 3 | 8,30 | 12.499 | 0,07 |

* 3 hours daily

Key competences are necessary for all citizens for personal fulfilment, active citizenship, social inclusion, and employability in a knowledge society. As a result, it is a fundamental task of the school system to assist children and adolescents in their development of key competences. In a European project, eight partners have developed strategies for changing mathematics education in order to contribute to this objective and have put these concepts into practice.

This book illustrates results of the project „KeyCoMath – Developing Key Competences by Mathematics Education".

- Concepts for improving mathematics education with the aim of developing key competences are suggested.
- Proved strategies for initial and in-service teacher education, which foster competence-oriented attitudes and beliefs of university students and teachers, are presented.
- As examples, ten learning environments for competence-oriented mathematics education, which were developed by teachers or university students and tested in school, are given.

www.KeyCoMath.eu

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Lifelong Learning Programme

Developing Key Competences by Mathematics Education