ASSESSING MATHEMATICALLY CHALLENGING PROBLEMS

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The aim of this article was twofold. First, to propose a model for assessing mathematical challenging problems and secondly to investigate the abilities of a group of in-service teachers to propose mathematically challenging problems based on the model suggested. The results indicated that mathematical challenging tasks may be characterised as those that are cognitively demanding and also require from students to develop other key competences: digital, social, communication in mother tongue, learning to learn and sense of initiative. About half of the participants of this study were able to provide mathematically cognitively demanding tasks which encompassed at least three of the other key competences. Participants appeared to find most difficult to incorporate in their mathematical tasks the "digital competence" and "learning to learn".

Keywords: Challenging tasks, key competences

INTRODUCTION

Mathematical tasks are considered to be in the core of student mathematics learning because they "convey messages about what mathematics is and what doing mathematics entails" (NCTM, 1991, p. 24). Different types of tasks may potentially influence students learning, thinking and understanding of mathematics in a different way (Henningsen & Stein, 1997; Kilpatrick, Swafford, & Findell, 2001). For low cognitive demand mathematics tasks, the emphasis is on practicing and repetition of known facts and procedures. In contrast, high cognitive/ challenging mathematics tasks require understanding and extending concepts (Hsu, 2013). In recent years the emphasis of mathematics education has turned towards teaching challenging cognitive demanding tasks, given that "Challenge is not only an important component of the learning process but also a vital skill for life" (Taylor, 2006, p. 2). International organizations stressed the importance of empowering students with key competences in order to be able to confront future challenges (e.g. European Parliament and Council, 2006).

The Professional Standards for Teaching Mathematics (NCTM, 1991) claimed that students' learning depends to a great extent on the way that teachers develop and implement mathematical tasks in their instruction. Therefore, it is important for teachers to gain a coherent understanding of the importance of mathematical challenge in teaching and learning mathematics, and moreover to be able to choose, design and implement such tasks in their teaching (Applebaum & Leikin, 2007). The purpose of the present study is twofold; first, to propose a mathematically challenging problem assessment tool, and second, to examine in-service teachers' ability to design challenging mathematical problems.

THEORETICAL CONSIDERATIONS

Defining mathematical challenging tasks

The main characteristic of mathematical challenging tasks is the fact that the solver is not immediately aware of the procedures or algorithms that are critical for its solution (Applebaum & Leikin, 2007; Powell, Borge, Fioriti, Kondratieva, Koublanova, & Sukthankar, 2009). Therefore, solvers are required to attempt to find a solution based on their knowledge and understanding (Applebaum & Leikin, 2007; Powell et al., 2009). Moreover, Guberman and Leikin (2013) added other characteristics in the definition of mathematical challenge: the tasks should be neither too easy nor too difficult, and should engage students in meaningful scenarios that develop mathematical curiosity and motivate students to persevere with task completion.

The characteristics of challenging mathematical tasks make them suitable to cover a range of audiences and didactical situations (Powell et al., 2009). In particular, challenging mathematical tasks can be attempted successfully by students of various mathematical backgrounds for diagnostic purposes, for learning new concepts and procedures, for developing mathematical understanding, for formative and final assessment (Powell et al., 2009). Recent studies verified the importance of challenging mathematics tasks in teaching and learning mathematics. A number of researchers (Hiebert et al., 2005; Hsu, 2013; Powell et al., 2009; Silver, Mesa, Morris, Star, & Benken, 2009; Stein & Lane, 1996) underlined their potential to maintain curiosity, stimulate creativity, promote flexible thinking, encourage collaboration and exploration, allow communication, increase students' understanding, promote conceptual understanding of mathematics, develop problem solving and reasoning abilities.

Mathematical challenging tasks and key competences

The abovementioned characteristics are essential in a society that requires citizens to be flexible at workplace, to adapt quickly to constant changes in an increasingly interconnected world and to be innovative, productive and competitive (European Parliament and Council, 2006; Halász & Michel, 2011). Taking into consideration the declaration of the European Parliament and Council (2006) for adapting the educational systems to the demands of todays' society by empowering citizens with lifelong learning abilities, it could be supported that the design of challenging problems that develop these abilities are of great importance.

Contemporary documents made use of the term "key competences" to determine the cognitive elements, the functional aspects (involving technical skills) as well as the interpersonal attributes (social or organizational skills) and ethical values that are important for personal fulfillment, active citizenship and employability (Halász & Michel, 2011). In this framework the European Parliament and Council proposed eight key competences (KC), which are of equally importance and also are interrelated, as they are defined below (European Parliament and Council, 2006; Halász & Michel, 2011). (a) *Communication in the mother tongue and* *communication in foreign languages*: The ability to understand, express and interpret procedures, concepts, ideas, thoughts, and feelings, in both written and oral form is fundamental to human interaction, (b) *Mathematical competences*: The ability to develop and apply mathematical thinking in an attempt to understand situations, to find explanations and to solve a range of problems in everyday situations, (c) *Digital competences:* The confident and critical use of information and communication technologies for the execution of educational, vocational and everyday work as well as for leisure and communication, (d) *Learning to learn:* The ability to pursue and organize the learning procedure of an individual or group, taking into account their needs and difficulties, the available time and information and the given opportunities and restrictions, (e) *Social, civic and cultural competences:* These competences embrace personal, interpersonal and intercultural aspects which equip individuals to engage in active and democratic participation in social and working life, (f) *Sense of initiative and entrepreneurship:* Creativity, innovation and risk-taking are among the characteristics that assist individuals to materialize their objectives.

These competences are anticipated to be acquired both by students at the end of compulsory education, as well as by adults through a process of developing and updating their skills (European Parliament and Council, 2006). Thus initial education should offer all students the opportunities to develop key competences in a sufficient level that will equip them for adult and working life (European Parliament and Council, 2006).

Designing mathematical challenging tasks

Despite the importance of mathematical challenging tasks, research has shown that it is not easy for teachers to design and implement such tasks in mathematics classrooms (Henningsen & Stein, 1997; Silver et al., 2009). The first barrier lies on teachers' pedagogical and content knowledge to design challenging tasks (Applebaum & Leikin, 2007). Teachers' content knowledge determines their understanding of the essence of mathematical challenge, their knowledge of challenging mathematics and their ability to approach challenging tasks (Applebaum & Leikin, 2007). As for the pedagogical knowledge, teachers' knowledge of the way that students cope with challenging mathematics, as well as different approaches and learning setting to teaching challenging mathematics are included (Applebaum & Leikin, 2007). Secondly, teachers find it difficult to design tasks that have a rich mathematical content, either by incorporating different mathematics topics and ideas or by demanding high cognitive effort (Silver et al., 2009; Stigler & Hiebert, 1999). In particular, in a study conducted by Silver and colleagues (2009) 84% of the activities designed by teachers focused on a single mathematics topic area rather than on multiple topics. Only 1 out of 3 activities was classified as a high-demand task, since teachers had difficulties to incorporate requirements for inquiry or explanations.

An important dimension of teachers' work is to find and/or adapt tasks. Additionally, it is extremely important to evaluate whether a task is appropriate for a particular student, from various perspectives (level of difficulty, interest, prior knowledge)

(Guberman & Leikin, 2013). Moreover during employing the tasks in their instruction, teachers have to decide what they want students to achieve, what they need to emphasize, how to sequence the various activities and in what way to support students without reducing the challenge (NCTM, 1991; Vale & Pimentel, 2011). Hence, it is apparent that by providing teachers with ready-to-use challenging mathematical tasks is not sufficient for their implementation. Teachers need to be convinced about the importance of mathematical challenge and develop abilities to deal with such kind of mathematics (Applebaum & Leikin, 2007).

Another issue that appears to be surfacing is the definition of a “mathematical challenging task” in today’s society. Is it simply a task that offers a certain level of mathematical, cognitive challenge to students or should it encompass other competences? Thus, the aim of the study was twofold: first, to suggest a model for assessing mathematically challenging problems, and secondly, to investigate whether in-service teachers who participated in a post-graduate problem solving course could develop mathematical challenging problems that promote key competences.

METHODOLOGY

Participants, procedure and data collection

The research was conducted at the University of Cyprus with 29 post-graduate students (PGs), who were studying for an MA in Mathematics Education, during their Problem Solving Course. In this course ideas arising from the Research Project KeyCoMath were utilized. Twenty three of the PGs had a BA in Primary Education, and six had a BSc in Mathematics.

The postgraduate course on Problem Solving was organized in 13 three-hour seminars (a seminar per week). During the course students worked on the following topics: (a) Mathematical problems and problem solving: Definitions, stages and strategies in problem solving, different types of problems, factors that affect problem solving abilities, (b) Modelling Problems: description, modelling principles, modeling Prospective vs Problem Solving Perspective, assessment of Modelling, (c) Problems in international competitions (PISA and TIMSS), (d) Problem posing, (e) Teaching approaches for the development of problem solving skills, (f) Inquiry-based learning and problem-based learning, (g) Teachers role during problem solving, and (h) Key competences in mathematics education.

One of the assignments that the students had to do, for the fulfillment of the requirements of the course, was to work either individually or in groups (of 2 or 3 people) and develop four challenging mathematical problems which would also promote key competences. Moreover students had to identify how key competences can be developed through these challenging problems.

Data Analysis

In the present study we propose a methodological tool to examine the extent to which a problem could be considered as a challenging one. To do so, we synthesize different theoretical approaches. For instance, Silver and his colleagues (2009) adopted frameworks that were used to distinguish levels of demands in mathematical tasks (see Kilpatrick, Swafford & Findel, 2001) and proposed criteria for coding activities as high or low demand. In addition, a number of organizations (see European Parliament and Council, 2006) assert that students should develop key competences to meet contemporary society needs. Table 1, presents thoroughly the set of criteria that compose our proposed assessment tool. Based on the adopted frameworks, we propose that a problem could be classified as a challenging one if it required high mathematical cognitive demand and involved at least three out of the five examined key competences.

The first assessment criterion is concerned with the problem's potential to develop student's mathematical competence. Taking into consideration Silver's framework (2009), it was decided that a problem could be characterized as a "mathematical high cognitive demand", if it explicitly required students to explain, describe, justify, compare, make decisions, plan, formulate questions or be creative in some way (Silver et al., 2009). On the contrary, a "mathematical low cognitive demand" problem requires merely routine applications of known procedures, extremely guided structure or challenging on non-mathematical issues.

The second criterion of our framework relates to the extent to which a problem provides opportunities to develop student's digital competence. A problem would be classified as a high digital competence task if it explicitly required the use of a digital device to search, collect or analyze data, support critical thinking, creativity or innovation. No-use, unclear mention to the use of digital media or use of technology solely for executing computations were classified as low digital competence. The third key competence criterion involves social competence. A problem would be classified as high social competence task if required students' involvement in communicating with others, to work collaboratively, to understand, share and reinforce other's ideas. Problems that did not require group working or peer interaction were classified as low social competence tasks. Communication in mother tongue was interpreted as the ability to express and interpret concepts and present with clarity and accuracy their mathematical ideas. Thus, a problem would be classified as high in communicating in mother/mathematical tongue, if it asked students to present, justify or convince regarding their solution or explain the way in which they used mathematical language to interpret the problem. The key competence learning to learn was defined as student's awareness of his/her learning process and the ability to build on previous learning experiences to transfer knowledge in a new context. Thus, a high demand learning to learn task required the use of reflective tools, such as explicit description of the solution plan, or the extension of the proposed solution to a different context. Finally, the key competence initiative was conceptualized as student's initiative to take decisions, propose creative ideas, risk-taking in planning and managing steps in the solving procedure. In this sense, the assessment criterion for high initiative key competence included the engagement of students in taking decision, planning or evaluating situation, the existence of multiple solutions or solving plans. This would require students to turn their ideas into actions by judging the risk of each solution plan or by applying creative ideas. On the contrary, an ill-define problem or extremely guided one would be classified as low initiative.

Two researchers were independently assigned to rate each task using the abovementioned criteria. There was near unanimity in this coding, and wherever there was any disagreement this was discussed until consensus was reached.

Table 1: Description of assessment criteria

Mathematical competence	Digital competence	Social competence	Communication in mother tongue	Learning to learn	Initiative
			Low		
Routine applications, extremely guided, imposed solution	No-use, unclear mention	Absence of effective interaction, no tolerance to other ideas	No use of mathematical language, poor expression of ideas, insufficient data or without different forms of data (verbal, graphical, symbolic)	No use of reflective questions or potential of transferring knowledge	Ill-structure problems or extremely guided
			High		
Explain, describe, justify, make decisions, plan, analysis, investigate, explore	Explicit and effective use of digital device to search or analyze data	Constructive communicatio n, group working, respect to other ideas	Express and interpret concepts, thoughts and facts in oral and written form, proper use of mathematical language, flexible use of different representations	Reflective tools and questions, transfer of knowledge to new contexts	Initiative to take decisions, creativity, risk-taking, plan and manage solution

RESULTS

Table 2, presents the classification of the problems proposed by the students based on the assessment model described above. In particular, 37 out of 57 problems were classified as high demand mathematical cognitive competence. They required students to investigate real life situations, use and connect different mathematical concepts, processes and relationships. The majority of them were decision-making problems. Four problems required from students to decide which the best monthly payment plan was, based on the advertisement of four communication companies. To do so, it was required to mathematize the problem by (i) suggesting a mathematical model for evaluating the monthly cost based on the offers of the companies, (ii) proposing the cheaper package that meets specific needs or (iii) proposing the best package for a person based on the statements of his account. Another problem required from students to propose the cheaper heating option, by taking into consideration the dimensions of a flat. Another interesting problem involved finding the best wine list for a cellar shop based on real costs, promotion costs, sale costs, delivery costs and people preferences. Two groups of teachers used football scenarios. They asked students to predict the winner of World Cup 2014 in Brazil, by proposing a mathematical model that takes into consideration several parameters. A second group of high cognitive demand mathematical problems required the analysis of a situation and the design of a system for a specific goal. For instance, students were asked to design a camping place and a car park.

Twenty out of the 57 problems were classified as low cognitive demand because they simply asked students to apply known procedures. A number of low demand problems asked students to solve open problems, without providing adequate data or their questions did not involve mathematical procedures or calculation (e.g., to describe the recipe for a birthday cake).

Table 2: Classification of problems

	Math. compet.	Digital compet.	Social compet.	Communication in mother tongue	Learning to learn	Initiative	**Challenging Problems**
Low	20	46	4	29	37	22	30
High	37	11	53	28	20	35	27

The results of the study showed that only 11 mathematical problems gave opportunities to develop students' digital competence. In particular, eight of these problems asked students to analyse, find connections and evaluate data given in spreadsheets or in specific links. For example, a problem required students to use the data provided in a spreadsheet regarding the population of Cyprus and Japan and use the tools of the software to find which of the two countries has the greater aging population problem. Another problem asked students to search on internet about the data of runners participated in the Olympic Games in London and the World Championship Athletics 2014 and to propose the four best runners of the last years. The rest of the high digital competence problems required students to find the solution to the problem by using software. For instance, a problem asked students to use a dynamic geometry software to design the floor plan of a library, by taking into consideration the dimensions of furniture that were provided in a spreadsheet.

The majority of the proposed problems provided opportunities to develop students' social competence. In particular, 53 out of the 57 problems required students to work in groups, collaborate and communicate for the solution of the problems. Some problems required job assignment, so that all members could work constructively. Problems also included instructions regarding tolerance and respect to everybody's ideas. The four low social competence problems did not make any reference on group work or communication among group members.

Almost half of the problems (28) provided opportunities to develop communication in mother tongue. They required students to use accurate and concrete mathematical language, to express and explain their ideas both in written and oral form and to utilise different representations. For instance, a problem asked students to design a poster for presenting their solution to their peers. Other problems asked students to use graphs and convincing arguments to support their suggestion. The remaining 29 problems were classified as low social competence because they did not involve accurate use of mathematical language or utilisation of different representations.

Only 20 out of the 57 problems could promote “learning to learn”. In these problems participants included reflection tools (see Figure 1) or they required to extend the solution method in a new situation (e.g., after developing a model to select the most effective lights for University lecture theatre students were asked to extend their model to adapt in any room’s light). On the contrary, the rest of the problems did not include reflection tools or any extension questions. About half of the problems (35 out of the 57) provided opportunities to develop students’ sense of initiative. In particular, these problems required the construction of a mathematical model and the assignment of weights to the various criteria used in this model. For instance, a problem asked students to select the best three Universities for a perspective student based on specific criteria. To do so, it was required to assign weight to a set of criteria, such as the ranking of the university, the distance from home, the number of amenities, the distance from the airport and availability of free wi-fi. Twenty two problems were rated as low initiative, due to the fact that they were extremely structured or they were too open and ill-defined.

Work in groups to solve the following problem. It is important to listen carefully to the ideas of your peers and collaborate in a constructive manner.

A family searches for available flats to rent. As Mr. Antoniou mentioned they can afford a monthly rent up to 620 euro. Mr. Antoniou works in an accountancy firm, while Mrs. Antoniou is unemployed. They want to live in Latsia (area in Nicosia, Cyprus), since the children’s school is in this area. Use the data provided in the spreadsheet (dimensions, area, age, rent, etc.) and the map from Google earth to decide the best option for the family to rent. Explain your answer by using words, symbols, tables or graphs.

Once you have solved the problem, answer the following questions: (1) Write the mathematical concepts and processes you used in solving the problem, (2) Draw a diagram to show the changes in group thinking during the solution process, (3) Can the proposed model be used by another family? Explain, and (4) How difficult did you find this task? Explain.

Figure 1: An example of challenging mathematical problem.

Summing up, we concluded that only 27 out of the 57 problems could be classified as challenging (high mathematical cognitive demand and three out of the five key competences). It should be noted that ten problems met the criterion for high cognitive mathematical demand, but not the criterion of developing three out of the five key competences. Figure 1, presents a decision-making problem provided by one of the participants, which fulfilled all the criteria of our suggested model.

CONCLUSIONS

In this paper we proposed a model for assessing challenging mathematical problems based on mathematical features as well as other key competences. We suggested that a mathematical challenging problem should encompass high cognitive demand and at the same time enhance eat least three out of the following key competences: digital, social, communication in mother tongue, learning to learn and sense of initiative. This proposed model is in line with the definitions and characteristics of challenging mathematical problems suggested by other researchers (e.g., Hiebert et al., 2005; Hsu, 2013; Powell et al., 2009; Silver et al., 2009; Stein & Lane, 1996).

The postgraduate problem solving course allowed a group of in-service teachers to develop abilities to design challenging tasks up to a certain extent. Almost half of the proposed problems appeared to be able to develop highly mathematically cognitive tasks which incorporated at least three other key competences. However, it seemed that the in-service teachers of the study had a greater difficulty to develop mathematical challenging problems that promote "digital competence" and "learning to learn". Therefore, future studies should aim to investigate the way in which we may empower teachers to develop challenging mathematical problems which enhance students' key competences.

REFERENCES

Applebaum, M., & Leikin, R. (2007). Teachers' conceptions of mathematical challenge in school mathematics. In J. H. Woo, H. C. Lew, K. S. Park, & D. Y. Seo (Eds.), *Proceedings of the 31st International Conference for the Psychology of Mathematics Education* (Vol. 2, pp. 2-9). Seyoul, Korea: PME.

European Parliament and Council of the European Union (2006, December). *Recommendation of the European Parliament and of the Council of 18 December 2006, on key competencies for lifelong learning* (Official Journal L 394 of 30 December 2006). Retrieved from European Union website: http://eur-lex.europa.eu/LexUriServ/LexUriServ.do?uri=OJ:L:2006:394:0010:0018:EN:PDF

Guberman, R., & Leikin, R. (2013). Interesting and difficult mathematical problems: Changing teachers' views by employing multiple-solution tasks. *Journal of Mathematics Teacher Education*, *16*(1), 33-56.doi:10.1007/s10857-012-9210-7

Halász, G., & Michel, A. (2011). Key competences in Europe: Interpretation, policy formulation and implementation. *European Journal of Education*, *46*(3), 289-306. doi:0.1111/j.1465-3435.2011.01491.x

Henningsen, M., & Stein, M. K. (1997). Mathematical tasks and student cognition: Classroom-based factors that support and inhibit high-level mathematical thinking and reasoning. *Journal for Research in Mathematics Education*, *28*(5), 524-549.doi:10.2307/749690

Hiebert, J., Stigler, J. W., Jacobs, J. K., Givvin, K. B., Garnier, H., Smith, M., Gallimore, R. (2005). Mathematics teaching in the United States today (and tomorrow): Results from the TIMSS 1999 video study. *Educational Evaluation and Policy Analysis, 27*(2), 111-132. doi:10.3102/01623737027002111

Hsu, W. M. (2013). Examining the types of mathematical tasks used to explore the mathematics instruction by elementary school teachers. *Creative Education*, *4*(6), 396-404. doi:10.4236/ce.2013.46056

Kilpatrick, J., Swafford, J., & Findell, B. (2001). The strands of mathematical proficiency. In J. Kilpatrick, J. Swafford, & B. Findell (Eds.), *Adding it up: Helping children learn mathematics* (pp. 115 – 155). Washington, DC: National Academy Press.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics*. Reston, VA: Author.

Powell, B., Borge, I. C., Fioriti, G. I., Kondratieva, M., Koublanova, E., & Sukthankar, N. (2009). Challenging tasks and mathematics learning. In E. J. Barbeau & P. J. Taylor (Eds.), *Challenging mathematics in and beyond the classroom: The 16th ICMI study* (Vol. 12, pp.133-170). New York, NY: Springer.

Silver, E. A., Mesa, V. M., Morris, K. A., Star, J. R., & Benken, B. M. (2009). Teaching mathematics for understanding: An analysis of lessons submitted by teachers seeking NBPTS certification. *American Educational Research Journal*, *46*(2), 501-531. doi:10.3102/0002831208326559

Stein, M. K., & Lane, S. (1996). Instructional tasks and the development of student capacity to think and reason: An analysis of the relationship between teaching and learning in a reform mathematics project. *Educational Research and Evaluation*, *2*(1), 50-80.doi:10.1080/1380361960020103

Stigler, J. W., & Hiebert, J. (1999). *The teaching gap: Best ideas from the worlds' teachers for improving education in the classroom*. New York, NY: Free Press.

Taylor, E. W. (2006). The challenge of teaching for change. *New directions for adult and continuing education*, *2006*(109), 91-95. doi:10.1002/ace.211

Vale, I., & Pimentel, T. (2011). Mathematical challenging tasks in elementary grades. In M. Pytlak, T. Rowland, & E. Swoboda (Eds.), *Proceedings of the Seventh Congress of the European Society for Research in Mathematics Education* (pp.1154-1164). Rzeszow, Poland: CERME.

Acknowledgement

The project "KeyCoMath - Developing Key Competences by Mathematics Education" has been co-funded by the Lifelong Learning Programme of the European Union. This communication reflects only the author's view. The European Commission and the EACEA cannot be held responsible for any use which may be made of the information contained therein.