Developing by Key Competences Mathematics Education

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Key Competences for Lifelong Learning and Mathematics Education: Concepts for Teaching and Learning Mathematics in Class

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1 Introduction

This chapter aims to present a proposed framework which shows how basic skills and transversal competences can be developed by mathematics education. To this end, section two presents the Key Competences for Lifelong Learning according to the European Parliament and the Council of the European Union (2006). Section three provides the mathematical procedures and practices that students should develop according to various mathematics education organizations and researchers. Section four, having as a springboard the two previous sections, discusses a proposed framework which supports the development of students' key competences in mathematics. Finally, in section five we present assessment methods for students' key competences in mathematics.

2 Key Competences for Lifelong Learning

In the last ten years a lot of attention has been given at the European level to key competences for lifelong learning. A number of reports documented the importance of developing key competences at school in Europe (see European Commission/EACEA/Eurydice, 2012; Otten & Ohana, 2009). In addition, a significant number of research programs were conducted in the frame of the key competences (e.g., Q4i project: Quality for innovation in European schools, KeyCoNet project: Key Competence Network on School Education, EU-27: Key Competences and Teacher Education). At the end of 2006, these competences were defined by the European Parliament and the Council of the European Union (*Recommendation 2006/962/EC*), after a five year collaborative work between specialists to find a common framework of key competences for lifelong learning at the European level. Pepper (2011) characterized the development of these competences as 'an important policy imperative for EU member states' (Pepper, 2011, p. 335).
European Parliament and the Council of the European Union (2006) defined key competences as a multifunctional and transferable package of knowledge, skills and attitudes, which 'all individuals need for personal fulfillment and development, active citizenship, social inclusion and employment' (p. 13). In particular, European Parliament and the Council of the European Union (2006) identified eight key competences:

(1) Communication in the mother tongue
(2) Communication in foreign languages
(3) Mathematical competence and basic competences in science and technology
(4) Digital competence
(5) Learning to learn
(6) Social and civic competences
(7) Sense of initiative and entrepreneurship
(8) Cultural awareness and expression

These competences are equally important and interrelated. They are anticipated to be acquired both by students at the end of compulsory education, as well as by adults through a process of developing and updating their skills (European Parliament and Council of the European Union, 2006). Thus, initial education should offer all students the opportunities to develop key competences in a sufficient level that will equip them for adult and working life (European Parliament and Council of the European Union, 2006). In the following section, we provide a short description of the eight key competences:

"*Communication in the mother tongue*" requires an individual to have knowledge of words, terms, functional grammar and the functions of mother language. At the same time, this competence is connected to the acquisition of an individual's cognitive ability to interpret the world and relate to others. Moreover, it involves 'the skills to communicate both orally and in writing in a variety of communicative situations and to monitor and adapt their own communication to the requirements of the situation' (European Parliament and Council of the European Union, 2006, p. 14). This competence also includes the acquisition of positive attitude to critical and constructive dialogue, an acceptance of aesthetic qualities accompanied with an enthusiasm to try for them, and an interest in interaction with others (European Parliament and Council of the European Union, 2006; European Communities, 2007).

"*Communication in foreign languages*" requires an individual to have knowledge of the words, terms, functional grammar and structure of foreign languages. It includes skills for communication in foreign languages. These skills are comprehension of spoken messages, making conversations, reading and producing writing. Moreover, this competence involves 'the appreciation of cultural diversity, and an interest and curiosity in languages and intercultural communication' (European Parliament and Council of the European Union, 2006, p. 15).

"*Mathematical competence and basic competences in science and technology*" involves two essential competences: mathematics competence and science and technology competence.
'Knowledge in mathematics involves a sound knowledge of numbers, measures and structures, basic operations and basic mathematical presentations, an understanding of mathematical terms and concepts, and an awareness of the questions to which mathematics can offer answers. An individual should have the skills to apply basic mathematical principles and processes in everyday contexts at home and work, and to follow and assess chains of arguments. An individual should be able to reason mathematically, understand mathematical proof and communicate in mathematical language, and to use appropriate aids' (European Parliament and Council of the European Union, 2006, p. 15).
For science and technology competence, 'essential knowledge comprises the basic principles of the natural world, fundamental scientific concepts, principles and methods, technology and technological products and processes, as well as an understanding of the impact of science and technology on the natural world. Skills include the ability to use and handle technological tools and machines as well as scientific data to achieve a goal or to reach an evidence-based decision or conclusion' (European Parliament and Council of the European Union, 2006, p. 15).

"*Digital competence*" requires an individual to have knowledge of the 'nature, role and opportunities of technology in everyday contexts' (European Parliament and Council of the European Union, 2006, p. 16). This competence, also, includes 'the ability to search, collect and process information and use it in a critical and systematic way, assessing relevance and distinguishing the real from the virtual while recognizing the links' (European Parliament and Council of the European Union, 2006, p. 16).

"*Learning to learn*" requires 'an individual to know and understand his/her preferred learning strategies, the strengths and weaknesses of his/her skills and qualifications, and to be able to search for the education and training opportunities and guidance and/or support available. Learning to learn skills require firstly the acquisition of the fundamental basic skills such as literacy, numeracy and ICT skills that are necessary for further learning. Building on these skills, an individual should be able to access, gain, process and assimilate new knowledge and skills. Individuals should be able to organize their own learning, evaluate their own work, and to seek advice, information and support when appropriate. A positive attitude includes the motivation and confidence to pursue and succeed at learning throughout one's life' (European Parliament and Council of the European Union, 2006, p. 16).

"*Social and civic competences*" includes two essential competences: social competence and civic competence.
'Social competence involves knowledge of basic concepts relating to individuals, groups, work organizations, gender equality and non-discrimination, society and culture. The core skills of this competence include the ability to communicate constructively in different environments, to show tolerance, express and understand different viewpoints, to negotiate with the ability to create confidence, and to feel empathy.
Civic competence is based on knowledge of the concepts of democracy, justice, equality, citizenship, and civil rights, including how they are expressed in the Charter of Fundamental Rights of

the European Union and international declarations and how they are applied by various institutions at the local, regional, national, European and international levels. Skills for civic competence relate to the ability to engage effectively with others in the public domain, and to display solidarity and interest in solving problems affecting the local and wider community. This involves critical and creative reflection and constructive participation in community or neighborhood activities as well as decision-making at all levels, from local to national and European level, in particular through voting' (European Parliament and Council of the European Union, 2006, p. 17).

"*Sense of initiative and entrepreneurship*" requires individual to be able to turn ideas into action and to assess and take risks to materialize objectives. Individuals should also 'be aware of the ethical position of enterprises, and how they can be a force for good' (European Parliament and Council of the European Union, 2006, p. 17). In addition to this, 'an entrepreneurial attitude is characterized by initiative, pro-activity, independence and innovation in personal and social life, as much as at work' (European Parliament and Council of the European Union, 2006, p. 18).

Finally, "*Cultural awareness and expression*" 'includes an awareness of local, national and European cultural heritage and their place in the world. Skills relate to both appreciation and expression: the appreciation and enjoyment of works of art and performances as well as self-expression through a variety of media using one's innate capacities. Skills include also the ability to relate one's own creative and expressive points of view to the opinions of others and to identify and realize social and economic opportunities in cultural activity. A solid understanding of one's own culture and a sense of identity can be the basis for an open attitude towards and respect for diversity of cultural expression' (European Parliament and Council of the European Union, 2006, p. 18).

3 Mathematical Practices

In the mathematics education domain various organizations and educational authorities defined important mathematical procedures and practices that students should develop (National Council of Teachers of Mathematics, 2000; Standards for Mathematical Practice of the Common Core State Standards Initiative, 2011). In addition, the debate regarding the practical perspective and the mathematical rigor in developing mathematical curricular has been extensively discussed in various countries (Sullivan, 2011). In this framework, Ernest (2010) described the goals of the practical perspective and explained the importance of learning the mathematics adequate for general employment and functioning in society and mathematics necessary for various professional and industry groups. Furthermore, the Shape of the Australian Curriculum: Mathematics (Commonwealth of Australia, 2009) distinguished the practical and the specialized aspects of the mathematics curriculum and emphasized the need to educate students to be active, to interpret the world mathematically and use mathematics to make predictions as well as to take decisions regarding personal and financial priorities. The Victorian Department of Education and Early Childhood Development of the Government in Australia (2009) underlined the importance of developing strategic skills, such as knowing that mathematics might help, adapting mathematics to the context, knowing how accurate to be, and knowing if the result makes sense in context. It also made explicit twelve scaffolding practices that are appropriate to explore/make explicit what is known, challenge/extend students' mathematical thinking or demonstrate the use of a mathematical instrument. In particular, the twelve scaffolding practices refer to excavating, modeling, collaborating, guiding, convincing, noticing, focusing, probing, orienting, reflecting, extending and apprenticing.

Classroom mathematical practices focus on the taken-as-shared ways of reasoning, arguing, and symbolizing established while discussing particular mathematical ideas (Cobb, Stephan, McClain, & Gravemeijer, 2011). Mathematical practices describe the conditions under which students learn mathematics with deep conceptual understanding. The mathematical practices are not skill-based content that students can learn through direct teaching methods, but emerge over time from opportunities and experiences provided in mathematics classrooms (Hull, Balka, & Harbin-Miles, 2011). Thus, these opportunities and experiences should be coordinated by mathematics teachers and include challenging problems, collaborative groups and interactive discourse. The practices are interdependent, in other words are not developed in isolation from one another, and hence mathematics educators need to continually assess student progress on these practices in a holistic fashion. The intent is that these essential mathematical habits of mind and action pervade the curriculum and pedagogy of mathematics, in age-appropriate ways.

The Common Core State Standards for Mathematics described both Content Standards and Standards for Mathematical Practice. The Standards for Mathematical Practice of the Common Core State Standards Initiative (2011) described varieties of expertise that mathematics educators at all levels should seek to develop in their students. The theoretical foundation of the mathematical practices lies on important "processes and proficiencies" with longstanding importance in mathematics education. First, the mathematical practices are related to the NCTM (2000) process standards of problem solving, reasoning and proof, communication, representation, and connections. In addition, they take into consideration the mathematical proficiency of the National Research Council's report *Adding It Up*: adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately), and productive disposition. The following mathematical practices describe in a more mathematical-oriented framework the majority of the Australian scaffolding practices.

Make sense of problems and persevere in solving them.

The first practice involves the discussion, explanation and solution of a problem with multiple representations and in multiple ways, as well as students' persistence during the solution process. Students should be able to analyze givens, constraints, relationships, and goals, make conjectures about the solution and plan a solution, monitor and self evaluate their progress and change course if necessary. Thus, this practice consists of the following: working to make sense of a problem, making a plan, trying different approaches when the problem is difficult, solving a problem in more than one way, checking whether the solution makes sense and connecting mathematical ideas and representations to one another.

Reason abstractly and quantitatively.

The second practice relates to converting situations into symbols to appropriately solve problems as well as convert symbols into meaningful situations. In other words, it involves the appropriate use, interpretation and exploitation of the mathematical language. Thus, two salient abilities are involved; *decontextualizing* – to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, and *contextualizing*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Hence, this practice consists of the following: (a) representing problems and situations mathematically with numbers, words, pictures, symbols, gestures, tables and graphs, and (b) explaining the meaning of the numbers, words, pictures, symbols, gestures, tables and graphs.

Construct viable arguments and critique the reasoning of others.

The third practice involves justifying and explaining, with accurate language and vocabulary, why a solution is correct and comparing and contrasting various solution strategies and explaining the reasoning of others. Students may construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Students should be able to listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments. Summing up, this practice consists of explaining both what to do and why it works and working to make sense of others' mathematical thinking.

Model with mathematics.

Model with mathematics describes the ability to use a variety of models, symbolic representations and digital tools to demonstrate a solution to a problem, by identifying important quantities in a practical situation and mapping their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. In addition, students should be able to analyze those relationships mathematically to draw conclusions. Thus, this practice consists of applying mathematical ideas to real-world situations and using mathematical models such as graphs, drawings, tables, symbols, and diagrams to solve problems.

Use appropriate tools strategically.

This practice involves students' ability to combine various tools, including digital tools, to explore and solve problems as well as justifying their tool selection. Tools are used to deepen students' understanding of concepts. Students should be able to make decision regarding the appropriateness of the used tool. Thus, this practice consists of choosing appropriate tools for a problem and using mathematical tools correctly and efficiently.

Attend to precision.
Attend to precision relates to students' appropriate use of symbols, vocabulary and labeling to effectively communicate and exchange ideas. Students should be able to use clear definitions with others and in their own reasoning. This practice consists of communicating mathematical thinking clearly and precisely, and being accurate when someone counts, measures and calculates.

Look for and make use of structure.
This practice involves students' ability to see complex and complicated mathematical expressions as component parts. They can conceptualize complicated things, as single objects or as being composed of several objects. This practice consists of finding, extending, analyzing and creating patterns and using patterns and structures to solve problems.

Look for and express regularity in repeated reasoning.
This practice defines students' ability to discover deep, underlying relationships, find and explain subtle patterns and look for regularity in problem structures when solving mathematical tasks. This practice consists of using patterns and structures to create and explain rules and shortcuts, using properties, rules and shortcuts to solve problems and reflecting before, during and after solving a problem.

The above mathematical practices had a significant impact in the mathematics education community. For instance, the Mathematics Assessment Project, collaboration between the University of California, Berkeley and Shall Center at the University of Nottingham, developed an assessment material that brings into life and contributes in implementing in real classrooms the Common Core State Standards for Mathematical Practice.

4 Proposed Framework: Development of Concepts for Teaching and Learning in Class

4.1 Mathematics Education and Key Competences

Despite the importance of Key Competences and the great emphasis to their development at European level, little attention has been given to the role of mathematics education, as a main subject in the school system, in developing students' key competences. Based on this, the KeyCoMath project underlined that there is a need to infuse activities that promote the development of key competences in mathematics teaching and learning. Thus, the KeyCoMath project tackles the challenge of implementing the "European Reference Framework of Key Competences" (European Parliament and Council of the European Union, 2006) in mathematics education and focuses on the development, implementation and evaluation of the majority of the European key competences in mathematics. Specifically, at the beginning of the project, the following key competences were defined and elaborated in the framework of mathematics teaching and learning.

Mathematical competence. KeyCoMath argues that students' mathematical thinking can be developed and enhanced through an active, exploratory learning in open and rich situations.

Communication in the mother tongue. KeyCoMath closely intertwines doing mathematics and communicating with others orally or in written form. For this, KeyCoMath claims that students should be encouraged to talk about mathematics, to discuss ideas, to write down thoughts and reflections and to present results.

Digital competence. KeyCoMath supports that students should be engaged in learning environments that utilize digital media (e.g. spreadsheets, dynamic geometry, computer algebra). By working mathematically they develop a confident, critical and reflective attitude towards ICT.

Learning to learn. KeyCoMath recommends that emphasis should be given to students' self-regulated and autonomous learning. Thus, students should develop abilities to manage their learning (both individually and in groups), to evaluate their work and to seek advice and support when appropriate.

Social competences. KeyCoMath argues that teaching of mathematics should provide students opportunities to collaborate and communicate. They cooperatively do mathematics, discuss ideas, present findings and have to understand different viewpoints to achieve common mathematical results.

Sense of initiative. KeyCoMath supports that students should be encouraged to be creative, proactive and to turn ideas into action though exploratory, inquiry-based learning in mathematics.

Through this process students might develop abilities to plan, organize, and manage their work.

4.2 The Proposed Framework

In the following, we outline the way in which the KeyCoMath project investigated all the above Key Competences and Mathematical practices and reached a new framework for Key Competences that should be developed in the mathematics classroom. The purpose of the proposed framework is to synthesize the two strands of research in the domain of key competences, i.e., the European framework of key competences and the recent efforts in mathematics education to explicitly define important mathematical practices. Thus, in an attempt to provide a practical framework that defines mathematical practices which contribute in enhancing students' key competences, we suggest a five-pillar comprehensive description of mathematical practices and appropriate didactical approaches.

Explore problems and persevere in solving them.
We set off from the principle that one learns a lot of things in mathematics if he/she is given the opportunity to solve a lot of problems. Working on problems teaches students the importance of persistence when dealing with a mathematical problem and the importance to develop reasonable arguments. Through problem solving students learn to learn.
Through the exploration of problems students may also develop a sense of initiative since different approaches may be used. Students may be asked to solve a problem or carry out an assignment where limited or no specific directions are given. Such scenarios, force students to take their own initiative.

Approaches

- Use exploration activities: Mathematics discipline is a good environment for exploring. This gives students the opportunity to explore and learn by themselves.
- Offer plethora of problems: Students should be given the opportunity to solve different types of problems, for instance, easy and hard problems, problems that can be solved if you are working backwards. They should try to understand the meaning of problems, to analyze information, to identify the entry point for their solution.
- Engage students with problems with different solutions.
- Offer problems which can be solved with different approaches.
- Use open problems.
- Ask students to make conjectures: Students should try to make conjectures and plan approaches for solutions.
- Give students time: It is important to stress that learning to learn needs time, and time needs to be allocated to this purpose. Students should be given the time to reflect on activities and learn by themselves.
- Ask students to reflect on activities that they carried out: Through problem solving students should be asked to reflect on their own learning, monitor and evaluate their procedures and ask themselves whether their solution makes sense.

Communicate in mother tongue, construct viable arguments and critique the reasoning of others.
One of the main concerns of mathematics teaching should be the flexible transition from everyday language to the language used in mathematics:

- Each child has to have the possibility to express his/her own thoughts and feelings through a mathematical approach.
- It is a long journey to reach formal mathematics. This may start from early years when students are asked to add for example 3+2. In such cases students may be asked to tell a story which represents this mathematical sentence. Teachers should try to break the separation that exists between natural language and mathematics.
- When using mathematical language, students need to realize that they need to be precise and accurate. The aim will be to bring precision in their language. This is of great essence and necessary when students make a transition to more typical and formal mathematics.
- When referring to communication it needs to be stressed that this should be achieved both in written and spoken language.
- Communication in mother tongue should be used in the classroom for students to communicate and understand their teacher as well as their peers.

Approaches

- Ask students to produce a learning journal: Students may be asked to write a journal

about the mathematics they learn in the classroom.
- Require from students to produce a poster or presentation: Students should be given the opportunity to present their mathematical approaches, solutions, projects to others in the form of a presentation or a poster. Of course, such activities will also enhance students' social and communication skills as well as their mathematical skills.
- Engage students to activities that require mathematization of everyday life: Such activities aim to connect mathematical language to mother tongue and everyday life. Through the process of mathematization students are asked to represent a real-world situation symbolically (in mathematics language) by describing, identifying, formulating and visualizing the mathematical problem in their own way; and then moving back by making sense of the mathematical solution in terms of the real solution, including the limitations of the solution.
- Ask students to reformulate a mathematical idea: Students are asked to express the same mathematical idea in different ways.
- Offer students the possibility to work on projects: Students may develop communication in their mother tongue if they are working on their own projects.
- Ask students to communicate and defend mathematical ideas with the use of language, symbols, diagrams, actions.
- Ask students to listen and read carefully other people's arguments.
- Ask questions to clarify issues presented to students.
- Ask students to decide whether certain arguments make sense to them.

One may think that communication in a foreign language may be something that does not have a place in the mathematics classroom. However, this is far from the truth. Foreign language may be a way of describing the communication in another language which students are not familiar with. This key competence is necessary to be developed when students are asked to communicate with a programming language.

Approaches
- Learning to communicate with the programming language: Students may be asked to write or read something in programming language. Of course this may be done in various levels of education and in various levels of difficulty.

Social competence may be developed through the interaction and communication that students have in the classroom both with their peers and their teacher.

Approaches
- Ask students to listen, talk, write, understand, and communicate: Students should be given the opportunity to communicate in verbal and written form. They should learn how to listen carefully to other, to talk with precision and make themselves understood by others.
- Require from students to accept different solutions and respect arguments: Students must learn to respect the opinions and arguments provided by other people and also accept different solutions.
- Engage students in co-actions when communicating.
- Require from students accuracy in their communication: Insist on accuracy in various activities.
- Apply dialogic learning in classrooms: Teaching should involve the approach of dialogic learning.

Model with mathematics, contextualizing and decontextualizing everyday situations.

Students should be able to use mathematics to deal with problems that arise in everyday life, society and workplace. They should be able to model, contextualize and decontextualize everyday situations. For instance, in early years this may start with simply stating an equation to solve a mathematical problem while later students may have to construct more complex models to solve a problem which may involve situations such as decision making, system analysis and trouble shooting.

Approaches
- Ask students to apply their knowledge in real world problems.
- Require from students to make sense of quantities and relationships in problems.
- Ask students to use different types of models, representations, graphs, diagrams to represent various relationships among quantities, concepts, shapes. Create coherent representations that facilitate to solve a problem.

- Offer students the possibility to contextualize and decontextualize mathematical ideas. Be able to represent concrete situations symbolically and be able to understand and deal with them and the reverse, be able to understand symbols and translate them into real life concrete scenarios.
- Offer activities to students in order to become able to understand and translate abstract ideas into symbols and become able to use them.

Use appropriate tools strategically.

Students with mathematical proficiency should be able to make decisions regarding which tool they need or is appropriate to carry out a specific task. Tools may be simply paper and pencil, rulers, protractors, calculators, mathematics software. Nowadays, the ability to use digital tools strategically is one of the central competences that individuals need to develop.

Digital competence allows students to develop intuitions about various mathematical concepts. Students need to know and also use various mathematical programs. Furthermore, when referring to digital competence it is important to raise the issue that students need to learn how various machines work.

Approaches

- Offer students the possibility to decide which tools to use and how to use them, for example, tools, such as paper and pencil, calculator, ruler, digital technologies, protractor, compass.
- Train students to use any kind of software system: Asking students to work with any kind of software system often increases and improves students' mathematical understanding. This may allow students to better visualize, compare and predict results.
- Offer students the possibility to use modern software systems: Students must be given the opportunity to work with modern software systems.
- Engage students with virtual reality activities in mathematics: Students must also be given the opportunity to work in developing mathematical ideas with the use of virtual reality.

Look for and make use of structure and generalizations, attend to precision.

Students should be able to identify a pattern or structure in what they see in mathematics and also be able to find commonalities and reach generalizations or find shortcuts in procedures. In the process of doing these as well as in the products of their mathematical activities students should be able to communicate with precision and accuracy.

Approaches

- Ask students to look for patterns or structure.
- Require from students to recognize that quantities can be represented in different ways.
- Give students the opportunity to use patterns or structures to solve problems.
- Offer students the possibility to view complicated concepts either as single objects or part of compositions of several objects.
- Give students tasks where they need to notice repeated patterns, calculations or methods and look for general methods or shortcuts.
- Ask students to reflect and evaluate the reasonableness of results and make generalizations.
- Require from students to communicate precisely and with accuracy.
- Ask students to state the meaning of concepts, symbols, and carefully specify units of measure.
- Require from students to calculate accurately and efficiently, to carefully state explanations and provide accurate labels.

5 Key Competences and Assessment

In recent years great emphasis is given on the ways used to assess key competences. Gordon et al. (2009) found that the most of the EU member states implemented curricula that include competences and identified assessment as one of their most important components. In particular, assessment might give information about learning process, can lead to the development of key competences and may support consequently effective changes (European Commission, 2012). Indeed, 'The assessment of key competences or similar learning outcomes that emphasize not only knowledge but also skills and attitudes in relation to contexts intended as preparation for lifelong learning' (European Commission, 2012 p. 4). Despite the importance of the assessment, the European Commission/EACEA/Eurydice report (2012), declares that most countries use national tests to assess only three out of the eight key competences (communication in the mother tongue, communication in foreign languages and mathematical competence and basic competences in science and technology). In addition to this, the national assessments in the most countries focus on the basic skills (mother tongue and mathematics) and in some of them focus on the science, foreign languages and social and civic competences (European Commission/EACEA/ Eurydice, 2012). For this, there is a need to reveal assessment methods for the key competences.

The first step towards assessment is the operationalization of the key competences, as it has been done in previous sections (Gordon et al., 2009). By defining the key competences and the scope of the assessment, as well by discussing the learning outcomes might give an insight for the assessment criteria (Sadler, 1987; Wolf, 2001). Furthermore, by describing the characteristics of the assessment tasks and the analysis of their results in the learning process, students and teachers can identify and appraise performances.

KeyCoMath project proposed four characteristics that tasks for measuring students key competences should have:

(1) include important mathematical aspects without complexity,
(2) allow the emergence of different solutions or processes,
(3) encourage and request pupils' productions (drawings, reasons),
(4) demand reflections such as descriptions, explanations or reasons.

First, as the aim of the assessment tasks is to reveal solver's cognitive processes and results, a suitable task should be clearly specified, easy to understand, is comprehensibly formulated and it fits to the current learners' state of knowledge. A similar conception is mentioned by Harlen (2007) 'A clear definition of the domain being assessed is required, as is adherence to it' (p. 18).
Secondly, although the task is based on known facts it might offer opportunities to the solver to link various competencies, to think critically and to deal actively in a situation. The exploration of open situations offers opportunities to the solver to provide a variety of solutions or different ways of approaches and problem-solving strategies. These type of tasks allows pupils to work on their own pace and abilities, to develop mathematical competences on their individual level, forcing them to activate their initiative and autonomy.
Thirdly, the use of appropriate scenarios such as occupational and social contexts asks learners to operate with everyday life problems and situations and adapt their knowledge in different set of circumstances (European Centre for the Development of Vocational Training, 2010). Such contexts demands from the solver to exploit their social, civic and cultural awareness.
Finally, appropriate assessment task should encourage the actual use and level of language, enhancing the competence of communication. As Gallin and Ruf (1998) mentioned, the use of language enables thoughts to be clarified and helps a response to be elicited.

Other assessment methods with the potential to assess key competences are among others standardised tests, attitudinal questionnaires, performance-based assessment, and portfolio assessment, teacher, peer and self-assessment practices (Looney, 2011; Pepper, 2013), as follow:

- *Standardised tests*: they mainly used for the assessment of some of the key competences (communication, mathematical competence, competences in science and technology), due to the fact that these three competences can be directly linked to individual subjects (Eurydice, 2009). As the other competences are more general and are related to several subjects, difficult appropriate standardised test might be developed (Pepper, 2013).

- *Attitudinal questionnaires*: these questionnaires aimed at capture learners' attitudes for learning to learn and generally for assessing affective and metaffective domains. However there is a difficulty of separating cognitive and affective aspects of learning (Frederiksson & Hoskins, 2008). Additionally a discrepancy between what students answer and what they actually do appears (European Commission, 2012).

- *Performance-based assessment*: this type of assessment includes portfolios, reflective diaries, experiments, group work, play, presentations, interviews and role plays (Looney, 2011). The observation of such behaviors might provide reliable results. 'Variation in teachers' judgements within and between schools is nonetheless a risk…' (Pepper, 2013, p. 18).

Classroom observation and dialogue is considered as an appropriate method for the assessment of key competences, in contrast to questionnaires and tests (Pepper, 2013). An inquiry based learning environment might offers great opportunities to this direction (Gallin, 2012). Through students experimentation in combination with teachers' involvement several key competences may be revealed and consequently might be assessed. For instance, as students attempt to find one or more solutions to a given task, mathematical competence is inevitably engaged. A careful selection of a task might also engage the cultural awareness and expression competence as the given information or even the required ones might be connected with the social and economic opportunities in cultural activity.

Additionally, if a teacher provides students with digital and electronic means as facilitators of problem solving process digital competence may also be developed. The use of digital means, in combination with cooperative work, forces learners to depict the communication and social competences.

Exchanging thoughts among the learners, discussing ideas and solutions between teacher and students, keeping track of their thoughts, problems and findings, judging others solutions, advising their peers might develop and assess the competences of communication and learning to learn.

Moreover, the role of teachers is important to provide appropriate feedback on remarkable insights. Reflection and redirection of students' thoughts may enable them to be aware of their strengths and weaknesses and consequently to identify fruitful use of them (Gallin & Ruf, 1998).

6 Conclusions

Summing up, the importance of key competences for lifelong learning has been widely documented (e.g. Eurydice, 2012; Otten & Ohana, 2009). Based on the previous sections of this chapter, is obvious that mathematics education may achieve the goal of developing key competence in a sufficient level and thus be able to build a more qualified citizen for the needs of today's world. Citizens who are able to explore everyday problems with perseverance, who are able to communicate and collaborate effectively, construct viable arguments and judge the reasoning of others, who develop models, who are able to use tools appropriately and who look for and use structures and generalizations, are sufficiently equipped to confront contemporary challenges.

Besides the development of key competence, their assessment is of equal importance. The development and assessment of key competences should be directed linked in order to gain better results in education. In particular, by revealing the ways that lead to the enhancement of key competences, insights are given for the development of appropriate assessment criteria (Sadler, 1987; Wolf, 2001). Additionally, the assessment process might give information about the learning process, and thus lead to the development of key competences (European Commission, 2012).

Hence, the research program KeyCoMath contributed in two strands: first, by defining the key competences and by proposing appropriate didactical approaches for their development; secondly, by proposing assessment methods with the potential to assess key competences. In order to ensure that "key competences" are in the focus point of both researchers and educators, it is important to educate teachers on this topic and especially train them about the ways in which key competences can be interwoven with their curriculum. Therefore, further studies could focus on developing didactical concepts that aim to maximize the impact of key competences in the field of teacher education and/or ways to measure the effectiveness of such programs.

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